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COMBINATORIAL PROOFS OF SOME ENUMERATION IDENTITIES

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We give a combinatorial proof of the following :

Theorem—The number of compositions of n is the same as the number of self-conjugate partitions with largest part equal to n.

Corollary—The number of n-reflected lattice paths equal twice the number of compositions of n. Some other enumeration identities are also proved.

1. INTRODUCTION

It is easily verified that the number of self-conjugate partitions of n with largest equal to m is the coefficient of $t^m q^n$ in

$$\sum_{m=0}^{\infty} \frac{t^m q^{m^2}}{(1-tq^2)(1-tq^4)\ldots(1-tq^{2m})}.$$

Hence the generating function for all self-conjugate partitions with largest part equal to m is what we get when we set $q = 1$:

$$\sum_{m=0}^{\infty} \frac{t^m}{(1-t)^m} = \frac{1-t}{1-2t}$$

or

$$\sum_{m=1}^{\infty} \frac{t^m}{(1-t)^m} = \frac{t}{1-2t}. \qquad \ldots(1.1)$$

On the other hand, if $C_m(t)$ is the enumerating generating function for compositions with exactly m parts, then

$$C_m(t) = (t + t^2 + t^3 + \ldots)^m = t^m (1-t)^{-m}. \qquad \ldots(1.2)$$

And so the generating function with no restriction on the number or size of parts is

$$C(t) = \sum_{m=1}^{\infty} C_m(t) = \frac{t}{1-2t}. \qquad \ldots(1.3)$$

A comparison of (1.1) and (1.3) leads us to our main theorem.

Theorem—The number of composition of n is the same as the number of self-conjugate partitions with largest part equal to n.

The exact meaning of the theorem should be clear from the following example : There are eight compositions of 4, viz.

$$4, \quad 31, \quad 13, \quad 2^2, \quad 21^2, \quad 121, \quad 1^22, \quad 1^4.$$

There are also eight self-conjugate partitions with largest part 4, viz.

$$41^3, \quad 421^2, \quad 4321, \quad 43^21, \quad 4^22^2, \quad 4^232, \quad 4^33, \quad 4^4.$$

This paper is centered around a combinatorial proof of the main theorem. The n-reflected lattice paths of the corollary were recently studied by Agarwal and Andrews[1], and are defined as follows :

Definition—A lattice path from (0, 0) to (n, n) is said to be n-reflected if for each (x, y) in the path $(n - y, n - x)$ is also in the path. For example, there are four 2-reflected lattice paths, viz.,

(0, 0), (1, 0), (2, 0), (2, 1), (2, 2); (0, 0), (1, 0) (1, 1), (2, 1), (2,2);

(0, 0), (0, 1), (1, 1), (1, 2), (2,2); (0, 0), (0, 1), (0, 2), (1, 2), (2,2).

In the next section we shall prove some subsidiary theorems which are required in proving our main results combinatorially.

2. Subsidiary Theorems

Theorem 2.1—The number of compositions of n with exactly m parts equals the number of partitions into m distinct parts with largest part n.

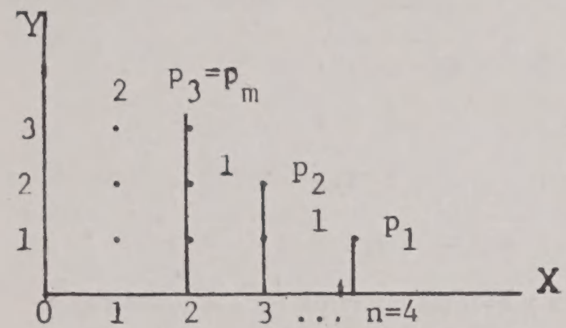
First proof (*by generating function*)—It is well known that the coefficient of $t^n q^N$ in

$$\frac{t^m q^{m(m+1)/2}}{(1-tq)(1-tq^2)\ldots(1-tq^m)}$$

is the number of the partitions of N into m distinct parts with largest part n. Hence the generating function for all the partitions into m distinct parts with largest part n is what we get when we set $q = 1$, viz.,

$$\frac{t^m}{(1 - t)^m},$$

and the theorem follows immediately by noting that this function also enumerates the compositions with exactly m parts [see eqn. (1.2)].



Second proof (combinatorial) —Let be the graph of a partition into m distinct parts with largest part n (In the above graph $n = 4$, $m = 3$; also note that the X-axis is drawn one unit of length below the last row and the Y-axis one unit of length to the left of the first column). We draw vertical lines from the corner point of each part and measure the distance of each line from its preceding one taking.

Y-axis also into consideration. We see that these distances give rise to a composition of n into exactly m parts. The correspondence being one-one the theorem is proved.

$4 + 3 + 2 \longrightarrow 211$

$4 + 3 + 1 \longrightarrow 121$

$4 + 2 + 1 \longrightarrow 112$

Theorem 2.2—The number of compositions of n is the same as the number of partitions into at most n distinct part with largest part n.

PROOF : Let $P(n, m)$ denote the number of partitions into m distinct parts with largest part n, and $C_{n,m}$ the number of compositions of n with exactly m parts. Then by Theorem 2.1, we have

$$C_{n,m} = P(n, m). \qquad \text{...(2.1)}$$

Equation (2.1) implies

$$\sum_{m=1}^{n} C_{n,m} = \sum_{m=1}^{n} P(n, m),$$

and the theorem follows immediately.

The one-to-one correspondence established to prove Theorem 2.1 holds good here too.

$4 \longleftrightarrow 4$

$4 + 3 \longleftrightarrow 3,1$

$4 + 2 \longleftrightarrow 2,2$

$4 + 1$	$\longleftrightarrow 1, 3$
$4 + 3 + 2$	$\longleftrightarrow 2, 1, 1$
$4 + 3 + 1$	$\longleftrightarrow 1, 2, 1$
$4 + 2 + 1$	$\longleftrightarrow 1, 1, 2$
$4 + 3 + 2 + 1$	$\longleftrightarrow 1, 1, 1, 1$

Now we shall establish a natural bijection for the following :

Theorem 2.3—The number of self-conjugate partitions with largest part n is the same as the number of partitions into at most n distinct parts with largest part n.

We first define a k-bend.

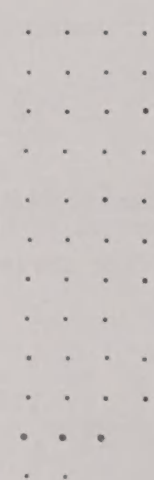
Definition —We call a right-bend ......, a k-bend if the number of dots in first row and first column are both equal to k. Thus by 1-bend we mean single dot., by 2-bend ... by 3-bend ..., etc.

Proof of the Theorem 2.3—Let $\pi = a_1 + a_2 + \ldots + a_r$ $(\eta = a_1 > a_2 > \ldots > a_r)$ be a partition

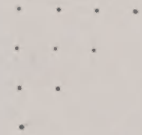
We consider a graph which consists of r successive bends viz., a_1-bend, a_2-bend,. ...a_r-bend. We see immediately that this graph represents a self-conjugate partition with largest part equal to n. The correspondence being one-to-one, the theorem is proved.

Example—Consider the case $n = 4$.

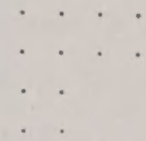
Partitions into at most 4 distinct parts with largest part 4	Mapping	Self-conjugate partition with largest part 4
$4 + 3 + 2 + 1$	(4-bend) + (3-bend) + (2-bend) + (1-bend). $\longleftrightarrow$	
$4 + 3 + 2$	(4-bend) + (3-bend) + (2-bend) $\longleftrightarrow$	
$4 + 3 + 1$	(4-bend) + (3-bend) + (1-bend) $\longleftrightarrow$	



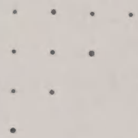
4 + 2 + 1 (4-bend) + (2-bend) + (1-bend)



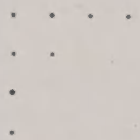
4 + 3 (4-bend) + (3-bend)



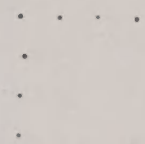
4 + 2 (4-bend) + (2-bend)



4 + 1 (4-bend) + (1-bend)



4 4-bend



3. Proofs of the Main Results

The bijections established to prove Theorems 2.2 and 2.3 together give rise a natural one-one onto mapping for our main theorem.

A bijection for the corollary can easily be obtained once we recall the following result due to Agarwal and Andrews[1].

Theorem 2.4—The n-reflected lattice paths are in one-to-one correspondence with the self-conjugate partitions with largest part $\leqq n$.

4. Conclusion

Since a subset of $\{1, 2, ..., n\}$ is also a partition into at most n distinct parts with largest part $\leqslant n$, the discussion of the preceding sections was a natural way to look at the following :

Theorem 4.1—Let A_n be the number of compositions of all integers $\leqq n$. B_n the number of self-conjugate partitions with largest part $\leqq n$. C_n the number of n-reflected lattice paths and D_n the number of subsets of $\{1, 2, ..., n\}$.

Then

$$A_n = B_n = C_n = D_n = 2^n.$$

Reference

1. A. K. Agarwal, and G. E. Andrews, *J. Stat. Pl. Inf.* **14**, (1986), 5-14.

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A GENERALIZED VARIATIONAL INEQUALITY INVOLVING A GRADIENT

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In this paper we study a generalized variational inequality problem related to a certain general mathematical programming problem. Some existence theorems are established. Consequently, we obtain existence results and give bounds for optimal solutions of the programming problem.

1. INTRODUCTION

Given a continuous map $F : R^n \to R^n$, a continuous linear function $\psi : R^n \to R$, and a nonempty set K in R^n, the problem of finding $\bar{x} \in K$ such that

$$< F(\bar{x}), x - \bar{x} > \geqslant \psi(\bar{x}) - \psi(x) \text{ for all } x \in K \qquad \ldots(1)$$

is called a variational inequality. Taking ψ as constant over the set K, we get a special case in which (1) becomes

$$< F(\bar{x}), x - \bar{x} > \geqslant 0 \text{ for all } x \in K. \qquad \ldots(2)$$

Theory as well as applications of the variational inequality have been well documented in the literature (see. e. g. Cottle *et al.*[3] and Kinderlehrer and Stampacchia[4]). In recent years, various extensions of this problem have been proposed and analyzed.

An important generalization of (2) is obtained by taking F to be a multifunction. In this case the problem is as follows: Given a multifunction V from R^n into itself, find vectors $\bar{x} \in K$ and $y^* \in V(\bar{x})$ such that

$$< y^*, x - \bar{x} > \geqslant 0 \text{ for all } x \in K. \qquad \ldots(3)$$

It is known as the generalized variational inequality (see Chan and Pang[2] and references therein).

Many problems in mathematical programming can be reduced to variational inequalities and thus can be studied in the framework of variational inequality problems.

In particular, we have the following : Let f be a continuously differentiable convex function in R^n. Then the problem of finding $\bar{x}$ in K such that

$$f(\bar{x}) = \min_{x \in K} f(x)$$

in equivalent to that of finding $\bar{x}$ in K satisfying the variational inequality

$$< \nabla f(\bar{x}), x - \bar{x} > \geqslant 0 \text{ for all } x \in K.$$

By using this fact, Mancino and Stampacchia[5] showed that their algorithm developed for the variational inequality (2) is available for approximating optimal solution for a convex programming problem.

Let $\varphi(x, y)$ be a real-valued function defined on $X \times D$, where X and D are closed, convex sets in R^n and R^p, respectively. Consider now a nonlinear programming problem (P), and a saddle point problem (SPP) given as

$$(P) \quad \min_{(x,y) \in U} \varphi(x, y),$$

where

$$U = \{(x, y) \in X \times D : \varphi(x, y) = \max_{\eta \in D} \varphi(x, \eta)\}$$

(SSP) : Find $x^0 \in X$, $y^0 \in D$ such that

$$\varphi(x^0, y) \leqslant \varphi(x^0, y^0) \leqslant \varphi(x, y^0)$$

for all $x \in X$ and $y \in D$. For a detailed study of these two problems, we see Mangasarian and Ponstein[7].

Let $\varphi(x, y)$ be differentiable with respect to x for every fixed y, and let $\nabla_1(\cdot, \cdot)$ denote the gradient of φ with respect to the first vector variable. The variational inequality problem that can be associated with (SPP) is as follows : Find $\bar{x} \in X$, $\bar{y}^* \in Y(\bar{x})$ such that

$$< \nabla_1 \varphi(\bar{x}, \bar{y}^*), x - \bar{x} > \geqslant 0 \text{ for all } x \in X \qquad \text{...(4)}$$

where

$$Y(x) = \{y^* \in D : \varphi(x, y^*) = \max_{\eta \in D} \varphi(x, \eta)\}. \qquad \text{...(5)}$$

It is easily seen that if $\varphi(g, y)$ is convex over X for every fixed $y \in D$, and $(\bar{x}, \bar{y}^*)$ is a solution of (4), then $(\bar{x}, \bar{y}^*)$ is also a solution of (SPP). Further, it follows from a lemma of Mangasarian and Ponstein[7] (Lemma 3.4, p. 509) that any solution $(\bar{x}, \bar{y})$ of (SPP) is an optimal solution of (P). Hence, the question of the existence of an optimal solution to (P) can be investigated via (4).

In this paper, we establish several existence theorems for (4) under various conditions on the map φ, each of which also guarantees the existence of an optimal solution of (P). Further, for (P), when it is solvable and φ is convex-concave over $X \times D$, we give simple bounds for all components of all optimal solutions.

2. Main Results

Given a multifunction V from R^n into itself, V is said to be upper semicontinuous if a sequence $\{x^n\}$ converging to x, and a sequence $\{y^n\}$, with $y^n \in V(x)^n$, converging to y, implies $y \in V(x)$. Under this definition, the point-to-set map $x \mapsto Y(x)$ as given by (5) is upper semi-continuous. Clearly, $Y(x)$ is a closed set. In what follows, it is assumed that $Y(x)$ is nonempty and bounded. Thus, for every $x \in X$, $Y(x)$ is a compact subset of D.

The following result which will be needed in the proof of Theorem 2 is due to Parida and Sen[10]. Here, for a set C in R^p, $P(C)$ denotes the collection of all compact convex subsets of C.

Theorem 1— Let S be a compact convex set in R^n, and C a closed convex set in R^p. Let $V: S \to P(C)$ be upper semicontinuous and $M: S \times C \to R^n$ be continuous. Then there exist $\bar{x} \in S, \bar{y} \in V(\bar{x})$ such that

$$<M(\bar{x}, \bar{y}), x - \bar{x}> \geqslant 0 \text{ for all } x \in S.$$

Theorem 2—Let $\varphi(x, y)$ be a continuous convex-concave function oxer $X \times D$, and let $\varphi(x, y)$ be continuously differentiable over X for every fixed $y \in D$. Further, assume that $Y(x)$ is nonempty and bounded. If there is a $\bar{u} \in X$ and a constant $r > \|\bar{u}\|$ such that

$$\max_{y^* \in Y(x)} <\nabla_1 \varphi(x, y^*), \bar{u} - x> \leqslant 0 \qquad \ldots(6)$$

for each $x \in X$ with $\|x\| = r$, then there exists a solution to (4), and hence an optimal solution to (P).

Proof : Let $X_r = \{x \in X : \|x\| \leqslant r\}$. Clearly, X_r is compact and convex. Since $\varphi(x, y)$ is concave in $y \in D$, $Y(x)$ is a compact and convex subset of D. It follows from Theorem 1 that there exist $\bar{x} \in X_r, \bar{y}^* \in Y(\bar{x})$ such that

$$<\nabla_1 \varphi(\bar{x}, \bar{y}^*), x - \bar{x}> \geqslant 0 \text{ for all } x \in X_r. \qquad \ldots(7)$$

We distinguish two cases.

Case 1—$\|\bar{x}\| < r$. Given $x \in X$, we can choose $0 < \lambda < 1$ small enough so that $w = \lambda x + (1 - \lambda)\bar{x}$ lies in X_r. Then it follows from (7) that

$$\begin{aligned} 0 &\leqslant <\nabla_1 \varphi(\bar{x}, \bar{y}^*), w - \bar{x}> \\ &= \lambda < \nabla_1 \varphi(\bar{x}, \bar{y}^*), x - \bar{x} > + (1 - \lambda) <\nabla_1 \varphi(\bar{x}, \bar{y}^*), \bar{x} - \bar{x}>, \\ &= \lambda<\nabla_1\varphi(\bar{x},\bar{y}^*), x - \bar{x}> \end{aligned}$$

and consequently, $(\bar{x}, \bar{y}^*)$ satisfies (4).

Case 2—$\|x\| = r$. Since $\bar{u} \in X_r$, it follows from (6) and (7) that $<\nabla_1 \varphi(\bar{x}, \bar{y}^*), \bar{u} - \bar{x}> = 0$. Since $\|\bar{u}\| < r$, for any given $x \in X$, again we can choose

$0 < \lambda < 1$ small enough so that $w' = \lambda x + (1 - \lambda)\bar{u}$ lies in X_r. Then, proceeding as in Case 1, it can be shown that $(\bar{x}, \bar{y}^*)$ satisfies (4). Therefore, we have

$$<\nabla_1 \varphi(\bar{x}, \bar{y}^*), x - \bar{x}> \geqslant 0 \text{ for all } x \in X. \qquad ...(8)$$

Now, from (5), (8) and the convexity of $\varphi(x, \bar{y}^*)$ over X, we get

$$\varphi(x, \bar{y}^*) \geqslant \varphi(\bar{x}, \bar{y}^*) \text{ for all } x \in X$$

$$\max_{\eta \in D} \varphi(\bar{x}, \eta) = \varphi(\bar{x}, \bar{y})^*,$$

which implies that $(\bar{x}, \bar{y}^*)$ solves (SPP). Finally, it follows from Lemma 3.4, p. 509 of Mangasarian and Ponstein[7] that $(\bar{x}, \bar{y}^*)$ is an optimal solution of (P).
which implies that $(\bar{x}, \bar{y}^*)$ is an optimal solution of (P).

The proof of Theorem 2 shows that the following result also holds.

Theorem 3—Let $\varphi(x, y)$ be a continuous convex-concave function over $X \times D$, and let $\varphi(x, y)$ be continuously differentiable over X for every fixed $y \in D$. Further, assume that $Y(x)$ is nonempty and bounded. If for some real number $r > 0$, the solution of the variational inequality

$$\bar{x} \in X_r, \bar{y}^* \in Y(\bar{x})$$

$$<\nabla_1 \varphi(\bar{x}, \bar{y}^*), x - \bar{x}> \geqslant 0 \text{ for all } x \in X_r$$

(which certainly has solutions) satisfies the estimate $\|\bar{x}\| < r$, then there exists a solution to (4), and hence an optimal solution to (P).

Now we consider some special mappings of interest in applications and show that condition (6) of Theorem 2 is satisfied for these mappings. The definitions we give extend the concept of nontonicity for single-valued functions (see Minty[8]) to multi-valued functions.

A multifunction $V : K \to R^n$ is said to be monotone on K if for every pair of vectors $x, u \in K$

$$<y^* - v^*, x - u> \geqslant 0 \text{ for all } y^* \in V(x), v^* \in V(u).$$

The map V is called strictly monotone if for every pair of vectors $x, u \in K, x \neq u$.

$$<y^* - v^*, x - u> 0 \text{ for all } y^* \in V(x), v^* \in V(u).$$

Proposition 1—Let $\varphi : R^n \times R^p \to R$ be continuously differentiable over X for every fixed $y \in D$. If $\varphi(x, y)$ is convex in $x \in X$, then the multifunction $x \mapsto \{\nabla_1 \varphi(x, y^*) : y^* \in Y(x)\}$ is monotone on X.

PROOF : From convexity of $\varphi(x, .)$ over X, we have

$$\varphi(x, v^*) - \varphi(u, v^*) \geqslant <\nabla_1 \varphi(u, v^*), x - u>$$

$$\varphi(u, y^*) - \varphi(x, y^*) \geqslant <\nabla_1 \varphi(x, y^*), u - x>$$

for every $y^* \in Y(x)$ and $v^* \in Y(u)$. By the definition of $Y(x)$, we have

$$\varphi(x, y^*) - \varphi(u, v^*) \geqslant \varphi(x, v^*) - \varphi(u, v^*)$$

$$\varphi(u, v^*) - \varphi(x, y^*) \geqslant \varphi(u, y^*) - \varphi(x, y^*).$$

Now, by adding the above four inequalities, it is seen that the map $x \mapsto \{\nabla_1 \varphi(x, y^*): y^* \in Y(x)\}$ is monotone on X.

If F is strictly monotone on K, then (2) has, at most, one solution. To see this, let both $\bar{x}$ and $\bar{y}$ be solutions to (2) with $\bar{x} \neq \bar{y}$. Then

$$\begin{aligned} 0 &< \langle F(\bar{x}) - F(\bar{y}), \bar{x} - \bar{y} \rangle \\ &= - \langle F(\bar{x}), \bar{y} - \bar{x} \rangle - \langle F(\bar{y}), \bar{x} - \bar{y} \rangle \\ &\leqslant 0 \end{aligned}$$

a contradiction. Using arguments similar to this and the conclusion of Proposition 1, it can be shown that (P) has, at most, one optimal solution if $\varphi(x, y)$ is strictly convex in $x \in X$ for every fixed $y \in D$.

In the following theorem, we take X to be a pointed convex cone. The polar of the cone X is given by

$$X^* = \{y \in R^n : \langle x,y \rangle \geqslant 0 \text{ for all } x \in X\}.$$

Theorem 4—Let X be a pointed closed convex cone in R^n, and let $\varphi(x, y)$ be as in Theorem 3. Suppose that $Y(x)$ is nonempty and bounded for every $x \in X$. If there is a $\bar{u} \in X$ and a $\bar{v}^* \in Y(\bar{u})$ such that $\nabla_1 \varphi(\bar{u}, \bar{v}^*) \in \operatorname{int} X^*$, then (4) has a solution, and hence, (P) has an optimal solution.

PROOF: We denote $p = \nabla_1 \varphi(\bar{u}, \bar{v}^*)$. Since $p \in \operatorname{int} X^*$, the set $B = \{x \in X : p^T(x - \bar{u}) \leqslant 0\}$ is compact, and consequently, for all $x \in X \setminus B$, $p^T(x - \bar{u}) > 0$. This implies that there is a constant $r > \|\bar{u}\|$ such that $p^T(x - \bar{u}) > 0$ for all $x \in X$ with $\|x\| = r$. From the convexity of φ and Proposition 1, it follows that the map $x \mapsto \{\nabla_1 \varphi(x, y^*) : y^* \in y(x)\}$ is monotone. Hence, for any $x \in X$ with $\|x\| = r$, we have

$$\begin{aligned} \langle \nabla_1 \varphi(x, y^*), x - \bar{u} \rangle &\geqslant \langle \nabla_1 \varphi(\bar{u}, \bar{v}^*), x - \bar{u} \rangle \\ &= p^T(x - \bar{u}) \\ &> 0 \end{aligned}$$

for all $y^* \in Y(x)$. This shows that condition (6) of Theorem 2 is satisfied. Therefore, (4) has a solution and consequently, (P) has an optimal solution.

Following the same reasoning used by Mangasarian and McLinden[8], and Parida and Sen[9] for providing numerical bounds for optiml solutions of convex programs we obtain the following result for (P).

Throrem 5—Let $X = R^n_+$ and let $\varphi(x, y)$ be a continuous convex-concave function over $R^n_+ \times D$. Further, let $\varphi(x, y)$ be continuously differentiable over R^n_+ for every fixed $y \in D$. Suppose that $Y(x)$ is nonempty and bounded for every $x \in R^n_+$. If there is a $\bar{u} \geqslant 0$ and a $\bar{v}^* \in Y(\bar{u})$ such that $\nabla_1 \varphi(\bar{u}, \bar{v}^*) > 0$, then there exists $(\bar{x}, \bar{y}^*)$ which solves (P). Any such solution is bounded as follows :

$$\|\bar{x}\|_1 \leqslant \langle \nabla_1 \varphi(\bar{u}, \bar{v}^*), \bar{u} \rangle / \min_{1 \leqslant i \leqslant n} \{\nabla_1 \varphi(\bar{u}, \bar{v}^*)\}_i. \qquad \ldots(9)$$

PROOF : The existence part follows from Theorem 4. To get the bound (9), let $(\bar{x}, \bar{y}^*)$ be an optimal solution of (P). Since $(\bar{x}, \bar{y}^*)$ is also a solution of (4), we have

$$\langle \nabla_1 \varphi(\bar{x}, \bar{y}^*), \bar{u} - \bar{x} \rangle \geqslant 0.$$

From the convexity of $\varphi(x, y)$ and Proposition 1, we have

$$\langle \nabla_1 \varphi(\bar{u}, \bar{v}^*) - \nabla_1 \varphi(\bar{x}, \bar{y}^*), \bar{u} - \bar{x} \rangle \geqslant 0.$$

Now, from these two inequalities, it follows that

$$\begin{aligned} \langle \nabla_1 \varphi(\bar{u}, \bar{v}^*), \bar{u} \rangle &\geqslant \langle \nabla_1 \varphi(\bar{u}, \bar{v}^*), \bar{x} \rangle \\ &\geqslant \min_{1 \leqslant i \leqslant n} \{\nabla_1 \varphi(\bar{u}, \bar{v}^*)\}_i . \sum_{1=i}^{n} \bar{x}_i \\ &= \min_{1 \leqslant i \leqslant n} \{\nabla_1 \varphi(\bar{u}, \bar{v}^*)\}_i . \|\bar{x}\|_1. \end{aligned}$$

This completes the proof of the theorem.

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CHARACTERIZATIONS OF EXTREMALLY DISCONNECTED SPACES

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In this note, the author obtains several characterizations of extremally disconnected spaces by utilizing preopen sets and semi-preopen sets.

1. INTRODUCTION

Quite recently, Sivaraj[9] has obtained some characterizations of extremally disconnected spaces by utilizing semi-open sets due to Levine[6]. Mashhour *et al.*[7] have defined preopen sets. Recently, Andrijević[1] has introduced a new class of sets, called semi-preopen sets, in topological spaces. The class of semi-preopen sets contains both the class of semi-open sets and the class of preopen sets. The purpose of this note is to obtain several characterizations of extremally disconnected spaces by utilizing preopen sets and semi-preopen sets.

2. PRELIMINARIES

Throughout the present note, by X we denote a topological space. Let A be a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}\,(A)$ and $\mathrm{Int}\,(A)$, respectively. A subset A is said to be preopen[7] if $A \subset \mathrm{Int}\,(\mathrm{Cl}\,(A))$. A subset A is called semi-open[6] (resp. semi-preopen[1]) if there exists an open (resp. preopen) set U of X such that $U \subset A \subset \mathrm{Cl}\,(U)$. Abd El-Monsef *et al.*[8] have introduced a weak form of open sets called β-open sets. The notion of β-open sets is equivalent to that of semi-preopen sets. In this note, we utilize the term "semi-preopen". The family of all preopen (resp. semi-open, semi-preopen) sets of X is denoted by $PO\,(X)$ (resp. $SO\,(X)$, $SPO\,(X)$). Levine[6] (resp. Andrijević[1]) showed that a subset A of X is semi-open (resp. semi-preopen) if and only if $A \subset \mathrm{Cl}\,(\mathrm{Int}\,(A))$ (resp. $A \subset \mathrm{Cl}\,(\mathrm{Int}\,(\mathrm{Cl}\,(A)))$). These characterizations will be frequently utilized in the sequel.

The complement of a semi-open set is called semi-closed[3]. The intersection of all semi-closed sets containing a subset A of X is called the semi-closure[3] of A and is denoted by $\mathrm{sCl}\,(A)$. It is obvious that $\mathrm{sCl}\,(A)$ is semi-closed. A point x of X is said to be a θ-adherent[10] (resp. δ-adherent) point of A if $A \cap \mathrm{Cl}\,(U) \neq \phi$ (resp. $A \cap \mathrm{Int}\,(\mathrm{Cl}\,(U)) \neq \phi$) for every open set U containing x. The set of all θ-adherent (resp. δ-adherent) points of A is called the θ-closure (resp. δ-closure) of A and is denoted by $\mathrm{Cl}_\theta\,(A)$ (resp. $\mathrm{Cl}_\delta\,(A)$).

The following lemma is shown by Janković[5] but we shall give an another proof.

Lemma 2.1[5]—If $A \in PO(X)$, then $\mathrm{Cl}(A) = \mathrm{Cl}_\delta(A) = \mathrm{Cl}_\theta(A)$.

PROOF : It is obvious that $\mathrm{Cl}(S) \subset \mathrm{Cl}_\delta(S) \subset \mathrm{Cl}_\theta(S)$ for every subset S of X. Thus, it remains to show that $\mathrm{Cl}_\theta(A) \subset \mathrm{Cl}(A)$. Assume that $x \not\in \mathrm{Cl}(A)$, then $U \cap A = \phi$ for some open set U containing x. We have $U \cap \mathrm{Cl}(A) = \phi$ and hence $\mathrm{Cl}(U) \cap \mathrm{Int}(\mathrm{Cl}(A)) = \phi$. Since $A \in PO(X)$, we obtain $\mathrm{Cl}(U) \cap A = \phi$. This shows that $x \in X - \mathrm{Cl}_\theta(A)$. Consequently, we have $\mathrm{Cl}_\theta(A) \subset \mathrm{Cl}(A)$ and $\mathrm{Cl}(A) = \mathrm{Cl}_\delta(A) = \mathrm{Cl}_\theta(A)$.

Sivaraj[9] showed that $\mathrm{Cl}(A) = \mathrm{Cl}_\delta(A)$ for every $A \in SO(X)$. The following lemma is an improvement of this result.

Lemma 2.2—If $A \in SPO(X)$, then $\mathrm{Cl}(A) = \mathrm{Cl}_\delta(A)$.

PROOF : We show that $\mathrm{Cl}_\delta(A) \subset \mathrm{Cl}(A)$ since the opposite inclusion is obvious. Assume that $x \in X - \mathrm{Cl}(A)$, then $U \cap A = \phi$ for some open set U containing x. Since U is open, $\mathrm{Cl}(U) \cap \mathrm{Int}(\mathrm{Cl}(A)) = \phi$ and hence we have $\mathrm{Int}(\mathrm{Cl}(U)) \cap \mathrm{Cl}(\mathrm{Int}\,\mathrm{Cl}(A))) = \phi$. Moreover, since $A \in SPO(X)$, we obtain $\mathrm{Int}(\mathrm{Cl}\,U)) \cap A = \phi$. This shows that $x \in X - \mathrm{Cl}_\delta(A)$. Therefore, we have $\mathrm{Cl}_\delta(A) \subset \mathrm{Cl}(A)$ and hence $\mathrm{Cl}(A) = \mathrm{Cl}_\delta(A)$.

3. CHARACTERIZATIONS OF EXTREMALLY DISCONNECTED SPACES

A topological space X is called extremally disconnected (briefly E. D.) if $\mathrm{Cl}(U)$ is open in X for every open set U of X, or, equivalently if every two disjoint open sets of X have disjoint closures.

Theorem 3.1—The following are equivalent for a space X.

(a) X is E. D.

(b) The closure of every semi-preopen set of X is open.

(c) The δ-closure of every semi-preopen set of X is open.

(d) The δ-closure of every preopen set of X is open.

(e) The β-closure of every preopen set of X is open.

(f) The closure of every preopen set of X is open.

PROOF : This follows immediately from Lemmas 2.1 and 2.2 since $\mathrm{Cl}(A) = \mathrm{Cl}\,\mathrm{Int}(\mathrm{Cl}(A)))$ for every $A \in SPO(X)$ and $PO(X) \subset SPO(X)$

Janković[5] showed that if a space X is E. D. then $s\mathrm{Cl}(A) = \mathrm{Cl}_\theta(A)$ for every $A \in PO(X) \cup SO(X)$. We show that the converse is also true.

Theorem 3.2—The following are equivalent for a space X.

(a) X is E. D.

(b) $\mathrm{sCl}(A) = \mathrm{Cl}_\theta(A)$ for every $A \in PO(X) \cup SO(X)$

(c) $\mathrm{sCl}(A) = \mathrm{Cl}(A)$ for every $A \in SPO(X)$.

(d) $\mathrm{sCl}(A) = \mathrm{Cl}_\delta(A)$ for every $A \in SPO(X)$.

PROOF : $(a) \Rightarrow (b)$: This is shown in Corollaries 4.5 and 4.6 of Janković[5] $(b) \Rightarrow (a)$: First, let A be any preopen set of X. By Proposition 2.7 of Janković[5] and Lemma 2.1, we have $\mathrm{Int}(\mathrm{Cl}(A)) = \mathrm{sCl}(A) = \mathrm{Cl}_\theta(A) = \mathrm{Cl}(A)$. Therefore, $\mathrm{Cl}(A)$ is open in X and hence it follows from Theorem 3.1 that X is E. D. Next, let A be any semi-open set of X. We have $\mathrm{sCl}(A) \subset \mathrm{Cl}(A) \subset \mathrm{Cl}_\theta(A) = \mathrm{sCl}(A)$ and hence $\mathrm{sCl}(A) = \mathrm{Cl}(A)$. Therefore, it follows from Theorem 2.1 of Sivaraj[9] that X is E. D.

$(a) \Rightarrow (c)$: It follows from Lemma 2.6 of Janković[5] that for every subset S of X, $\mathrm{Int}\,\mathrm{Cl}(S)) \subset \mathrm{sCl}(S) \subset \mathrm{Cl}(S)$. Since X is E. D., by Theorem 3.1, $\mathrm{Cl}(A)$ is open in X for every $A \subset SPO(X)$. Therefore, we have $\mathrm{sCl}(A) = \mathrm{Cl}(A)$ for every $A \in SPO(X)$. Therefore, we have $\mathrm{sCl}(A) = \mathrm{Cl}(A)$ for every $A \in SPO(X)$.

$(c) \Rightarrow (d)$: This is an immediate consequence of Lemma 2.2.

$(d) \Rightarrow (a)$: Let U, V be any disjoint open sets. Then we have $\mathrm{sCl}(U) \cap V = \phi$. Since $\mathrm{sCl}(U) \in SO(X)$, we have $\mathrm{sCl}(U) \cap \mathrm{sCl}(V) = \phi$. By Lemma 2.2, we obtain $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \phi$. This shows that X is E. D.

Theorem 3.3—The following are equivalent for a space X.

(a) X is E. D.

(b) If $A \in SPO(X)$, $B \in SO(X)$ and $A \cap B = \phi$, then $\mathrm{Cl}(A) \cap \mathrm{Cl}(B) = \phi$,

(c) If $A \in SPO(X)$, $B \in SO(X)$ and $A \cap B = \phi$, and $\mathrm{Cl}_\delta(A) \cap \mathrm{Cl}_\delta(B) = \phi$,

(d) If $A \in PO(X)$, $B \in SO(X)$ and $A \cap B = \phi$, then $\mathrm{Cl}_\theta(A) \cap \mathrm{Cl}_\delta(B) = \phi$,

(e) If $A \in PO(X)$, $B \in SO(X)$ and $A \cap B = \phi$, then $\mathrm{Cl}(A) \cap \mathrm{Cl}(B) = \phi$.

PROOF : $(a) \Rightarrow (b)$: Suppose that $A \in SPO(X)$, $B \in SO(X)$ and $A \cap B = \phi$. We have $\mathrm{Cl}(A) \cap \mathrm{Int}(B) = \phi$. By Theorem 3.1, $\mathrm{Cl}(A)$ is open in X and hence $\mathrm{Cl}(A) \cap \mathrm{Cl}(B) = \mathrm{Cl}(A) \cap \mathrm{Cl}(\mathrm{Int}(B)) = \phi$ since $B \in SO(X)$.

$(b) \Rightarrow (c)$, $(c) \Rightarrow (d)$ and $(d) \Rightarrow (e)$: These follow easily from Lemmas 2.1 and 2.2.

$(e) \Rightarrow (a)$: This is obvious since every open set is preopen and semi-open.

Remark 3.4 : Extremal disconnectedness of a topological space cannot be characterized by the statement that every two disjoint preopen sets have disjoint closures. There exists an E. D. space not satisfying the statement.

Example 3.5[2]—Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a, b\}\}$. Then the space (X, τ) is E. D., $\{a\}, \{b\} \in PO(X)$, and $\mathrm{Cl}(\{a\}) \cap \mathrm{Cl}(\{b\}) \neq \phi$.

Theorem 3.6—The following are equivalent for a space X.

(a) X is E. D.

(b) If $A \in SO(X)$ and $B \in SPO(X)$, then $\mathrm{Cl}(A) \cap \mathrm{Cl}(B) = \mathrm{Cl}(A \cap B)$.

(c) If $A \in SO(X)$ and $B \in SPO(X)$, then $A \cap B \in SPO(X)$.

PROOF : First, we recall the fact that if U is open in X then $U \cap \mathrm{Cl}(S) \subset \mathrm{Cl}(U \cap S)$ for every subset S of X.

(*a*) $\Rightarrow$ (*b*) : Let $A \in SO(X)$ and $B \in SPO(X)$. By Theorem 3.1, $\mathrm{Cl}(B)$ is open in X and we obtain

$$\mathrm{Cl}(A) \cap \mathrm{Cl}(B) = \mathrm{Cl}(\mathrm{Int}(A)) \cap \mathrm{Cl}(B) \subset \mathrm{Cl}(\mathrm{Int}(A) \cap \mathrm{Cl}(B)) \subset \mathrm{Cl}(A \cap B).$$

Therefore, we have $\mathrm{Cl}(A) \cap \mathrm{Cl}(B) = \mathrm{Cl}(A \cap B)$.

(*b*) $\Rightarrow$ (*c*) : Let $A \in SO(X)$ and $B \in SPO(X)$. Then, we have

$$A \cap B \subset \mathrm{Cl}(\mathrm{Int}(A)) \cap \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))) = \mathrm{Cl}(\mathrm{Int}(A) \cap \mathrm{Int}(\mathrm{Cl}(B))) = \mathrm{Cl}(\mathrm{Int}(A \cap \mathrm{Cl}(B))) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A) \cap \mathrm{Cl}(B))) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A \cap B))).$$

This shows that $A \cap B \in SPO(X)$.

(*c*) $\Rightarrow$(*a*) : It is enough to show $\mathrm{Cl}(A) \cap \mathrm{Cl}(B) = \mathrm{Cl}(A \cap B)$ for all open sets A and B. Let A and B be any open sets of X. Then, $\mathrm{Cl}(A)$ and $\mathrm{Cl}(B)$ are semi-open and hence $\mathrm{Cl}(A) \cap \mathrm{Cl}(B) \in SPO(X)$. Therefore, we have

$$\mathrm{Cl}(A) \cap \mathrm{Cl}(B) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A) \cap (B))) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Int}(\mathrm{Cl}(B))) \subset \mathrm{Cl}(\mathrm{Cl}(A) \cap \mathrm{Int}(\mathrm{Cl}(B))) \subseteq \mathrm{Cl}(A \cap \mathrm{Int}(\mathrm{Cl}\, B))) \subset \mathrm{Cl}(A \cap \mathrm{Cl}(B)) \subset \mathrm{Cl}(A \cap B).$$

Consequently, we obtain $\mathrm{Cl}(A) \cap \mathrm{Cl}(B) = \mathrm{Cl}(A \cap B)$.

Remark 3.7 : Jankovic[4] showed that a space X is E. D. if and only if $A \cap B \in SO(X)$ for every $A, B \in SO(X)$. However, neither $PO(X)$ nor $SPO(X)$ can substitute for $SO(X)$ in this result. Because, in the space (X, τ) of Example 3.5 (X, τ) is

E. D. and $\{a, c\}, \{b, c\} \in PO(X, \tau)$, however $\{c\} = \{a, c\} \cap \{b, c\}$ is not contained in $SPO(X, \tau)$.

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SPLITTINGS OF ABELIAN GROUPS BY INTEGERS

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A finite subset M of non-zero integers is said to split a finite abelian additive group G, if there exists a set S in G such that each non-zero element of G has a unique representation in the form $m\,s$, $m \in M$ and $s \in S$ and O has no such representation. For the set $S(k) = [1, 2, \cdots, k]$ for $k \leqslant 700$, except $k = 24, 60, 62, 84, 144, 171, 180, 264, 312, 420, 480, 665, 684$, it is proved that if $S(k)$ splits $C(r) \times G$ for a finite abelian group G and cyclic group $C(r)$ of order r, with $r \equiv 1 \pmod{k}$, then $S(k)$ splits $C(r)$. Stein asked this question for all k. Our result gives a partial answer to his question.

1. INTRODUCTION

Let M be a finite set of non-zero integers, called a multiplier set, and let G be a finite abelian additive group. Assume that there is a set S in G such that each non-zero element in G has a unique representation in the form $m.\,s$, $m \in M$, $s \in S$ and that O has no such representation. Then M is said to split G and S is called a splitting set. Splittings were first considered by Stein[4].

If $(m, |G|) = 1$ for each $m \in M$, the splitting is called non-singular. Otherwise it is called singular. If each prime that divides $|G|$ divides at least one $m \in M$, then the splitting is called purely singular (see Stein[5] for more details).

Hickerson has conjectured that if the set $S(k) = [1, 2, \ldots, k]$ splits a finite abelian group G purely singularly, then G must be either $C(1)$, $C(k+1)$, or $C(2k+1)$, where $C(r)$ denotes the cyclic group of order r.

Stein[5] observed that for $k = 2,3, 4,6,$ and 8 if the set $S(k)$ splits $G_1 = G \times C(r)$ for a finite abelian group G, then $S(k)$ splits $C(r)$ provided that $r \equiv 1 \pmod{k}$ and posed the question[5] (Question IV_{-1}) for other values of k. In this connection we prove:

Theorem 1—If $S(k)$ splits $G_1 = G \times C(r)$ where G is a finite abelian group, $r \equiv 1 \pmod{k}$, and k satisfies Hickerson's conjecture, then $S(k)$ splits $C(r)$.

Hickerson[3] has proved the conjecture for $k \leqslant 700$ with exceptions $k = 24, 60, 62, 84, 144, 171, 180, 264, 312, 420, 480, 665$, and 684. Thus in view of Theorem 1, Stein's assertion is true for the values of $k \leqslant 700$, except perhaps the ones listed above.

2. Some Known Results

For the proof of Theorem 1 we make use of the following results.

Lemma 1—Let $G - [0] = M$. S be a splitting of a finite abelian group G. Then there exist subgroups H and K of G such that

(i) $G = H \times K$

(ii) M splits H non-singularly

(iii) M splits K purely singularly.

Further H and K are uniquely determined by these conditions.

This result is due to Hickerson[2] (Theorem 1.2.5). The proof shows that if P denotes the set of prime divisors of $| G |$ which are relatively prime to all elements of M then the subgroup H is the direct product of p-sylow subgroups of G where $p \in P$ and K is the direct product of p-sylow subgroups of G for $p \notin P$.

The converse of the above is true, i. e. if H and K are finite abelian groups such that M splits H non-singularly and M splits K purely singularly then M splits $H \times K$. In fact the following more general result follows from Theorem of Hamaker and Stein[1].

Lemma 2—Let H be a subgroup of the finite abelian group G. Suppose M splits both H and G/H and that the splitting of H is non-singular. Then M splits G.

The following result, which is a consequence of Lemma 1, Lemma 2 and a results due to Hamaker and Stein[1] (Theorem 4) reduces the study of splittings of a finite abelian group G to the case of non-singular splittings of cyclic groups $C(p)$ of prime orders and the purely singular splittings.

Lemma 3—Let G be a finite abelian group, M a set of positive integers, P the set of prime divisors p of $| G |$ such that $(p, m) = 1$ for all $m \in M$ and K the subgroup of G obtained by taking the direct product of all the p-sylow subgroups of G for $p \notin P$. Then M splits G if and only if M splits $C(p)$ for all $p \in P$ and M splits K.

3. Proof of Theorem 1

Let $r = p_1^{\alpha_1} \dots p_s^{\alpha_s}\ p_1^{\beta_1}\ q_t^{\beta_t}$ be the prime factorization of r where the primes p_i, $1 \leqslant i \leqslant s$ are $\leqslant k$ and primes q_j ; $1 \leqslant j \leqslant t$ are $> k$.

Since $S(k)$ splits $C(r) \times G$, by Lemma 3, it splits $C(q_j)$ and hence $q_j \equiv 1 \pmod{k}$, for $1 \leqslant j \leqslant t$ so that

$$r_1 = p_1^{\alpha_1} \dots p_s^{\alpha_s} \equiv 1 \pmod{k}. \qquad \dots(1)$$

Let $G_1 = H_1 \times K_1$ where H_1 and K_1 are uniquely determined by Lemma 1 such that

$S(k)$ splits H_1 non-singularly and K_1 purely singularly. Since k satisfies Hickerson's conjecture it follows that

$$K_1 = C(1) \text{ or } C(k+1) \text{ or } C(2k+1). \qquad \dots(2)$$

Since K_1 is the direct product of the p-sylow subgroups of G_1, where p runs over all prime divisors of $|G_1| = r\,|G|$ with $p \leqslant k$, we have

$$r_1 \mid |K_1|. \qquad \dots(3)$$

It follows from (1), (2) and (3) that

either (i) $r_1 = 1$ or (ii) $K_1 = C(r_1)$.

Case (i)—In this case we have $r = q_1^{\beta_1} \dots q_t^{\beta_t}$ and since $S(k)$ splits $C(q_j)$, for $1 \leqslant j \leqslant t$, by Lemma 3, $S(k)$ splits $C(r)$.

Case (ii)— In this case we have $K_1 = C(r_1)$.

Let $r_2 = q_1^{\beta_1} \dots q_t^{\beta_t}$ so that $C(r) = K_1 \times C(r_2)$.

Now since $S(k)$ splits $C(r_2)$ non-singularly as in case (i) and $S(k)$ splits K_1 by Lemma 2 it follows that $S(k)$ splits $C(r)$. This completes the proof.

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COHOMOLOGY AND REDUCIBILITY OF REPRESENTATIONS OF SEMISIMPLE Γ-GRADED LIE ALGEBRAS

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The invariant forms, in particular, the bilinear Cartan-Killing form is considered. Casimir invariants are determined for nondegenerate ΓGLA and the cohomology of a wide class of representations of semisimple ΓGLA is shown to vanish, Finally, the applications to reducibility of representations are discussed.

INTRODUCTION

Cohomological techniques have played a significant role in describing anomalies and renormalisability in quantum field theory, e.g., it was recognised by Becehi, *et al.*[1] that chiral anomalies belong to nontrivial cohomological classes associated with a gauge invariant classical action. Trace anomalies[2] too are characterised by nontrivial cohomological classes for a system having local Weyl symmetry. It was noted by Stora[3] that renormalisability of quantum field theory is associated with the triviality of the second cohomology of the BRS algebras. We will specially address ourselves to the cohomology properties (algebraic) of graded Lie algebras in the context of their structure. The Z_2-graded Lie algebras (popularly known as super Lie algebras) and the classification of the classical Lie super algebras were systematically developed by several author especially Kac[4'7]. Attempts have been made to obtain the Iwasawa and Langlands decompositions of such super Lie algebras and the various parabolic subalgebras are determined[8]. It is hoped that Schmidt induction scheme could be carried out to obtain the various irreducible representations of Z_2-gradeed Lie algebras.

In a previous communication[9], we had introduced the cohomology of Z_2-graded Lie algebras and subsequently it was extended to generalised Lie algebras (Γ GLA), also called color superalgebras. It was shown that the cohomology of certain representations of a broad class of GLA is trivial[8]. In the present communication, we analyse the properties of invariant forms to obtain the cohomology properties of nondegenerate Γ GLA's. The results are then used to study the reducibility of finite dimensional representations of semisimple Γ GLA's. In so doing, we generalise our definition of the coboundary operator.

The plan of the parer is follows. In section2, we introduce the basic concepts of Γ GLA and relate the structure of a ΓGLA to that of a graded Lie algebra with com-

mutation facter $\epsilon(\alpha, \beta)$ instead of the usual $(-1)^{\alpha\beta}$. Following Backhouse[11] we define the invariant forms and establish their symmetry properties, in particular, we obtain the bilinear Cartan-Killing form $K(l, m)$. In section 3, we obtain the Casimir invariants of a nondegenerate ΓGLA defined as one in which det $K(l. m) \neq 0$[12,13]. We also introduce the notion of a Γ graded ideal and make a few remarks on the structure of a nondegenerete ΓGLA. We also show that all derivations of a nondegenrate ΓGLA are inner. It may be noted that the invariant forms and Casimir invariants constructed in these sections are important examples of more general invariant forms constructed earlier[14,15]. However, the results and the methods used are useful for subsequent computations and have been expatiated for the sake of completeness.

In section 4, we find the class of representations of semisimple GLA for which the cohomology is trivial, proving a generalised version of Whitehead's lemma. Finally in section 5, Weyl's reducibiliry theorem for Lie algebras is generalised to ΓGLA. The notation used throughout is same as in Mitra and Tripathy[10]. Also $\mathscr{F}$ throughout characterised a field of characteristic zero.

2. Resume and Invariant forms of ΓGLA

In this section, we begin by giving a short resume' of ΓGLA. We relate the structure of a ΓGLA to that of a GLA with commutation factor $\epsilon(\alpha, \beta)$ instead of the usual $(-1)^{\alpha\beta}$. Finally, we obtain the invariant forms of order n and demonstrate their invariance and symmetry properties.

(A) *Resume of* Γ *Graded Algebras*

The following definitions are well known from the theory of ΓGLA (ref. 7) and are given below for the sake of completeness.

Definition 1—A vector space V is said to be Γ graded if we are given a family $(V_\gamma)_{\gamma \in \Gamma}$ of subspaces of V such that V is the direct sum

$$V = \bigoplus_{\gamma \in \Gamma} V_\gamma, \qquad \ldots(2.1)$$

Γ being on Abelian group (see Definition 4). An element of V is said to be homogenous of degree $\gamma \in \Gamma$ if it is an element of V_γ. A subspace V' of V is said to be Γ graded if $V' = \bigoplus_{\gamma \in \Gamma} (V' \cap V_\gamma)$.

Definition 2—An algebra S is called Γ graded if is underlying vector space is Γ graded

$$S = \bigoplus_{V \in \Gamma} S_\gamma \qquad \ldots(2.2a)$$

and if $S_\alpha S_\beta \subset S_{\alpha+\beta} \; \forall \; \alpha, \beta \subset \Gamma$.

If S has a unit element e, it follows that $e \in S_0$. A subalgebra S' is said to be Γ graded if it is Γ graded as a subspace of S.

Definition 3—A commutation factor on Γ is a mapping $\epsilon : \Gamma \otimes \Gamma \rightarrow K$, K being a commutative field so that

$$(1) \quad \epsilon(\alpha, \beta) \in (\beta, \alpha) = 1 \qquad \ldots(2.3a)$$

$$(2) \quad \epsilon(\alpha, \beta + \gamma) = \epsilon(\alpha, \beta)\,\epsilon(\alpha, \gamma) \qquad \ldots(2.3b)$$

$$(3) \quad \epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \gamma)\,\epsilon(\beta, \gamma) \text{ for all } \alpha, \beta, \gamma \epsilon \Gamma. \qquad \ldots(2.3c)$$

Definition 4—Let Γ be an Abelian group and let ϵ be a commutation factor on Γ. A Γ graded Algebra

$$\mathbf{L} = \bigoplus_{\gamma \in \Gamma} \mathbf{L}_\gamma \qquad \ldots(2.4)$$

whose product mapping is denoted by an angle bracket (<>) is called a Γ graded $\in$ Lie Algebra, or simply a Γ graded Lie Algebra (ΓGLA) if the following properties are satisfied.

$$(1) \quad <A, B> = -\epsilon(\alpha, \beta) <B, A> \; (\epsilon \text{ skew symmetry}) \qquad \ldots(2.5a)$$

$$(2) \quad \epsilon(\gamma, \alpha) <A, <B, C>> + \text{cyclic permutations} = 0$$
$$(\epsilon \text{ Jacobi identity}) \qquad \ldots(2.5b)$$

for all $A \in \mathbf{L}_\alpha$, $B \in \mathbf{L}_\beta$, $C \in \mathbf{L}_\gamma$, $\forall\ \alpha, \beta, \gamma \in \Gamma$.

When Γ consists of the abelian group (0, 1) under the addition operation, and the commutation factor $\epsilon(\alpha, \beta)$ is chosen to be $(-1)^{\alpha\beta}$,, L is a Graded Lie Algebra (Lie superalgebra) and

$$<\mathbf{L}_0, \mathbf{L}_0> C\, \mathbf{L}_0, \quad <\mathbf{L}_0, \mathbf{L}_1> C\, \mathbf{L}_2, \quad <\mathbf{L}_1, \mathbf{L}_1> C\, \mathbf{L}_0 \qquad \ldots(2.6)$$

while the ϵ Jacobi identity reduces to the form

$$<<A, B>, C> = <A, <B, C>> - (-1)^{\alpha\beta} <B, <A, C>>. \qquad \ldots(2.7)$$

(B) *Structure of a ΓGLA*

Given a Γ GLA L, as in (2.4), let,

$$\Gamma_0 = \{\alpha \in \Gamma : \epsilon(\alpha, \alpha) = +1\} \qquad \ldots(2.8a)$$

$$\Gamma_1 = \{\alpha \in \Gamma : \epsilon(\alpha, \alpha) = -1\}. \qquad \ldots(2.8b)$$

Let

$$\mathbf{L}_0 = \bigoplus_{\alpha \in \Gamma_0} \mathbf{L}_\alpha \qquad \ldots(2.9a)$$

and

$$\mathbf{L}_1 = \bigoplus_{\alpha \in \Gamma_1} \mathbf{L}_\alpha. \qquad \ldots(2.9b)$$

Then we have,

Proposition 1

$$\langle L_0, L_0 \rangle \subset L_0, \quad \langle L_0, L_1 \rangle \subset L_1, \quad \langle L_1, L_1 \rangle \subset L_0. \qquad \text{...(2.10)}$$

PROOF : Let $\alpha, \beta \in \Gamma$. Then

$$\langle L_\alpha, L_\beta \rangle \; C \; L_{\alpha+\beta} \text{ for all } L_\alpha, L_\beta \in L.$$

It is easy to show that

$$\epsilon(\alpha+\beta, \alpha+\beta) = \epsilon(\alpha, \alpha)\, \epsilon(\beta, \beta) \qquad \text{...(2.11)}$$

from which (2.11) follows trivially.

The ϵ Jacobi identity may be cast in the following form :

$$\langle A, \langle B, C \rangle\rangle = \langle\langle A, B\rangle, C\rangle + \epsilon(\alpha, \beta) \langle B, \langle A, C\rangle\rangle \qquad \text{...(2.12)}$$

where

$$A \in L_\alpha, B \in L_\beta.$$

(2.10) and (2.12) give L a ΓGLA, the structure of a GLA with commutation factor $\epsilon(\alpha, \beta)$ instead of the usual $(-1)^{\alpha\beta}$. The results in Ref. 8 are now directly applicable and will be used in the next sub-section.

(C) *Invariant Forms*

Following Mitra and Tripathy[10], we construct invariant forms for L as follows. Let P_0 and P_1 be the orthogonal projections of L onto L_0 and L_1. We define the multilinear map

$$K_n : \underbrace{L \otimes L \ldots \otimes L}_{n \text{ times}} \to \mathscr{F}, \text{ a scalar field by}$$

$$K_n(l_1, l_2, .., l_n) = \operatorname{tr}[P_0(\operatorname{ad} l_1 \operatorname{ad} l_2, \ldots, \operatorname{ad} l_n) - P_1(\operatorname{ad} l_1 \operatorname{ad} l_2, \ldots, \operatorname{ad} l_n)] \qquad \text{...(2.13)}$$

for all $l_i \in L$, $i = 1, 2, \ldots, n$, adl being the adjoint action of l_i on $l \in L$. It is easily seen using structure contants rhat $K_n((l_1, l_2, \ldots, l_n) = 0$ unless $\sum_{i=1}^{n} | l_i | = 0$, ...(2.14)

$| l_i | \in \Gamma$ being the degree of homegeneity of $l_i \in L_0 \cup L_1$.

The following lemma is useful.

Lemma 1—(a) (i) If $l \in L_0$, $P_i \operatorname{ad} l = P_i \operatorname{ad} l\, P_i = \operatorname{ad} l\, P_i$ ...(2.15a)

(ii) If $l \in L_1$, $P_i \operatorname{ad} l = P_i \operatorname{ad} l\, P_j = \operatorname{ad} l\, P_j$

$i \neq j = 0, 1$ ...(215b)

(b) $\operatorname{ad} \langle l, m \rangle = \operatorname{ad} l \operatorname{ad} m - \in (| l | . | m |) \operatorname{ad} m \operatorname{ad} l$, ...(2.16)

PROOF : (a) The results (i) and (ii) are a direct consequence of (2.2) in Proposition 1 and the fact that P_0 (P_1) annihilate $\mathbf{L}_1$ ($\mathbf{L}_0$) respectively.

(b) This follows from eqn. (2.12).

We now establish the symmetry and invariance properties of K_n.

Theorem 1 (a) $K_n(l_1, l_2, \ldots, l_n) = \epsilon(|l_1|, |l_l|)\, K_n(l_2, \ldots, l_n\, l_l)$ ...(2.17)

(b) $K_n(l_1, l_2, \ldots l_n) = 0$ if the set $\{l_1, l_2, \ldots l_n\}$ contains an odd no, of elements from $\mathbf{L}_1$.

(c) $K_n(<l, l_1>, l_2, \ldots . l_n> + \sum_{i=2}^{n} \in(|l| \sum_{t=1}^{i-1} |l_t|) K_n(l_1, \ldots, <l, l_i>$

$\ldots, l_n) = 0.$ ...(2.18)

PROOF : Using Lemma (1a)

$$\text{tr}\,[P_l\, \text{ad}l_1\, \text{ad}l_2, \ldots, \text{ad}l_n)] = \text{tr}\,(P_l\,(\text{ad}l_1)\,(\text{ad}l_2, \ldots, \text{ad}l_n)]$$

$$= \text{tr}\,[\text{ad}l_1\, P_k\,(\text{ad}l_2, \ldots, \text{ad}l_n)]$$

$$= \text{tr}\,[P_k\,(\text{adl}_2, \ldots\, \text{ad}l_n\, \text{ad}l_2)],$$

where

$$k = i \quad \text{if } l_l \in \mathbf{L}_0,$$

$$k \neq i \text{ if } l_l \in \mathbf{L}_1.$$

(using the symmetry of the trace operation).

Hence

$$K_n(l_1, l_2, \ldots l_n) = \text{tr}\,[P_0\,(\text{ad}l_1, \ldots, \text{ad}l_n) - P_1\,(\text{ad}l_1 \ldots \text{ad}l_n)$$

$$= (-1)^{|l_i|}\ K_n(l_2, \ldots, l_n, l_1) \text{ if } l_1 \in L_{|l_i|}\ |l_l| = 0,1$$

$$= \epsilon(|l_1|, |l_1|)\, K_n(l_2, \ldots, l_n, l_1)$$

(b) $K_n(l_1, l_2, \ldots, l_n) = \epsilon(|l_1|, |l_1|) K_n(l_2, \ldots, l_n, l_1)$

$$= \prod_{i=1}^{n} \epsilon(|l_l|\ |l_l|)\, K_n(l_1, l_2, \ldots l_n). \qquad \ldots(2.19)$$

From (2.19), $K_n(l_1, l_2, \ldots, l_n) = 0$ if there are an odd no. of elements in the set $\{l_1, l_2, \ldots, l_n\}$ belonging to $\mathbf{L}_1$.

(c) Using Lemma(1*b*) and the properties of $\epsilon(\alpha, \beta)$ we have

$$\epsilon(|l|, \sum_{t=1}^{l-1} |l_t|) K_n(l_1, \ldots < l, l_l>, l_{l+1}, \ldots, l_n)$$

$$+ \epsilon(| l | , \sum_{t=1}^{i} | l_t |)K_n (l_1, \ldots, l_i <l, l_{i+1} >, \ldots l_n)$$

$$= \epsilon(| l | , \sum_{t=1}^{i-1} | l_t |) [K_{n+1} (l_1, \ldots, l, l_i \ldots, l_n)$$

$$- \epsilon (| l | , | l_i |) K_{n+1} (l_1, \ldots, l_i, l, l_{i+1}, \ldots, l_n)$$

$$+ \epsilon(| l | , l_i |) K_{n+1} (l_1, \ldots, l_i, l, l_{i+1}, \ldots, l_n)$$

$$- \epsilon \; | l | , | l_i | + | l_{i+1} |) K_{n+1} (l_1, \ldots, l, l_i l_{i+1}, \ldots, l_n)]$$

$$= \epsilon (| l | , \sum_{t=i}^{i-1} | l_t |) [K_{n+1} (l_1,\ldots, l, l_i, l_{i+1},\ldots, l_n)$$

$$- \epsilon (| l | , \sum_{t=1}^{i+1} | l_t |) K_{n+1} (l_1, \ldots, l_i, l_{i+1}, l_1,\ldots, l_n)]. \qquad \ldots(2.20)$$

Using (2.20) we find that the terms in the sum in (2.19) cancel diagonally and the L. H. S. of (2.18) is

$$K_{n+1} (l, l_1, \ldots l_n) - \epsilon (| 1 | , \sum_{t=1}^{n} | l_t |) K_{n+1} (l_1, \ldots, l_n, l)$$

$$= K_{n+1} (l, l_1, \ldots, l_n) [l - \epsilon (| l | , | l | + \sum_{i=1}^{n} | l_t |)] \qquad \ldots(2.21)$$

(using (2.17)).

From (2.14), using $\epsilon (| l | ,0) = 1$, it is seen that (2.21) is identically zero, proving (2.18).

Thus far, we have established the existence of invariant forms of arbitrary degree (unless some of them vanish identically for a ΓGLA). In particular, there exists a bilinear form K_2 which we call the Cartan-Killing form K defined by

$$K (l, m) = K_2 (l, m). \qquad \ldots(2.22)$$

The symmetry properties of K are listed below :

(1) $K (l, m) = K (m, l)$ for all $l, m \in \mathbf{L}_0$ ...(2.23a)

(2) $K (l, m) = 0$ for all $l \in L_0, m \in \mathbf{L}_1$ ...(2.23b)

(3) $K (l, m) = - K (m, l)$ for all $l, m \in \mathbf{L}_1$ ...(2.23c)

$K (l, m)$ evidently combines a Riemannian with a symplectic structure being the direct sum of a symmetric form on $\mathbf{L}_0$ and a skew symmetric form on $\mathbf{L}_1$.

3. Casimir Invariants and Structure of a Semisimpme ΓGLA

We isolate the class of ΓGLA for which det $K(l, m) \neq 0$. These we call nondegenerate following Backhouse[11] and Pais and Rittenberg[12].

We now construct Casimir invariants for such ΓGLA. We choose a basis $\{l^\alpha\}$ for **L**. The dual basis $\{l_\alpha\}$ is chosen by the condition

$$K(l^\alpha, l_\beta) = \delta^\alpha_\beta,\ \alpha, \beta = 1, \ldots, t. \qquad \ldots(3.1)$$

This is possible as K is nondegenerate. We now show that

$$C_n = K_n(l_{\alpha_1}, \ldots, l_{\alpha_n})\, l^{\alpha}{}_n, \ldots, l^{\alpha}{}_1 \qquad \ldots(3.2)$$

is invariant under **L**. We will need the following leamma.

Lemma 2—(a) If $<l, l^\alpha> = L^\alpha_\beta\, l^\beta$ and $<l, l_\beta> = M^\alpha_\beta\, l_\alpha$

$$M^\alpha_\beta = -\epsilon(|l^\alpha|, |l|)\, L^\alpha \hat{\beta}. \qquad \ldots(3.3a)$$

(b) $|l^\alpha| = -|l_\alpha|$. ...(3.3b)

Proof : (a) $K(l^\alpha, <l, l_\beta>) = K(l^\alpha, M^\gamma_\beta\, l_\gamma) = M^\alpha_\beta$

$$K(<l, l^\alpha>, l_\beta = K(L^\alpha{}_\gamma\, l^\gamma, l_\beta) = L^\alpha{}_\beta.$$

Using Theorem 1*c* for $n = 2$, the result follows.

(b) $K(l^\alpha, l_\beta) = \delta^\alpha{}_\beta = 0$ unless $|l^\alpha| = -|l_\beta|$.

Hence $\epsilon(|l^\alpha| + |l_\alpha|, |l| = 1$, so that

(i) $\epsilon(|l^\alpha|, |l|) = \epsilon(|l|, |l_\alpha|)$ ...(3.4a)

(ii) $\epsilon(|l|, |l^\alpha|) = \epsilon(|l_\alpha|, |l|)$. ...(3.4b)

Invarianee of the Casimir operator C under **L** means that adl $C = 0$ for all $l \in$ **L** and $C \in U(L)$, the Universal Enveloping Algebra which consists of linear combinations of monomials of **L**. The bracket operation ad$l$ is defined by

$$\text{adl}\,(mn) = <l, mn> = <l, m>n - \epsilon(|l|, |m|)\, m <l, n>.$$

Theorem 2— $C_n = K_n(l_\alpha, \ldots, l_{\alpha_n})\, l^\alpha{}_n, \ldots, l^\alpha{}_1$ is invariant under **L**, where $|l|$ $|l_{\alpha_i}|$ $l^\alpha{}_i$, are the degrees of l, l_{α_i}, and $l^\alpha{}_i$ respectively, $i = 1, \ldots, n$.

Proof : $adl\ C_n = K_n(l_{\alpha_1} \ldots, l_{\alpha_n}) <l, l^\alpha{}_n> l^\alpha{}_{n-1}, \ldots l^\alpha{}_1$

$$+\sum_{i=2}^{n} \epsilon\,(\,|\,l\,|\,,\sum_{k=1}^{i-1}\,(l^{\alpha}{}_{n-k+1}\,|\,)\,K_n\,(l_{\alpha_1},\ldots,\,l_{\alpha_{n-i}}\,,\ldots\,l_{\alpha_i})\,l^{\alpha}{}_{n}$$

$$\ldots<l,l^{\alpha n-i+1}>\ldots,\,l^{\alpha}{}_{1}$$

$$=K_n\,(l_{\alpha_1},\,\ldots,\,l_{\alpha_n})\,L^{\alpha_n}_{\gamma_n}\,l^{\gamma_n}{}_{1}{}^{\alpha}{}_{n-1}|\ldots l^{\alpha}{}_{1}+\sum_{i=2}^{n}\epsilon\,(\,|\,l\,|\,,\,\sum_{k=1}^{i-1}\,|\,l^{\alpha}{}_{n-k+1})$$

$$\times\;K_n\,(l_{\alpha_1},\,\ldots,\,l_{\alpha n-i+1}\;\ldots\,l_{\alpha_n}\,)\times\,l^{\alpha}{}_{n}\,\ldots\;\mathrm{L}^{\alpha_{n-i+1}}_{\gamma_{n-i+1}}\;l^{\gamma}{}_{n-i+1}\,\ldots\,l^{\alpha}{}_{1}$$

$$(=l_{\alpha_{n-i+1}}) \qquad \ldots(3.5)$$

Using Lemma 2a, noting that

$$|\,l_{\gamma_{n-i+1}}\,|\;\|\;|\,l^{\alpha}{}_{n-i+2}\,|+|\,l\,|\,, \qquad \ldots(3.6)$$

and relabelling $l_{\gamma_{n-i+1}}$ as $l_{\alpha\;n-i+1}$, (3.5) becomes

$$\mathrm{adl}\;C_n=-\,[\,\sum_{i=1}^{n}\;\epsilon\;\;(\,|\,l\,|\,,\,\sum_{k=1}^{i}\,|\,l^{\alpha}{}_{n-k+1}\,|\,)\,\epsilon\,(\,|\,l\,|\,,\,|\,l\,|\,)\,K_n\,(l_{\alpha_1}\,\ldots,$$

$$<l,\;l_{\alpha_{n-i+1}}\;>\ldots l_{\alpha_n})]\times\,l^{\alpha}{}_{n}\ldots\,l^{\alpha}{}_{1}. \qquad \ldots(3.7)$$

Using (3.4) and extracting $-\,\epsilon\,(\,\sum_{k=1}^{n}\,|\,l_{\alpha_n}\,|\,,\,|\,l\,|\,)\,\epsilon\,(\,|\,l\,|\,,\,|\,l\,|\,)$

in (3.7) gives

$$[K_n\,(<l,\,l_{\alpha 1}>\ldots\,l_{\alpha_1})+\sum_{i=2}^{n}\;\epsilon\,(\,|\,l\,|\,,\sum_{k=1}^{i-1}\;\;|\,l_{\alpha_k}\,|\,)$$

$$\times\;\;K_n\,(l_{\alpha_1}\,\ldots\,<l,\,l_{\alpha_i}\,>,\ldots,\,l_{\alpha_n})]\,l^{\alpha}{}_{n},\,\ldots,\,l^{\alpha}{}_{1}$$

which is zero by Theorem lc. This proves the result.

We now make a few remarks on the structure of a nondegenerate ΓGLA which will be used in the next section.

Definition 1—A linear subset N of a Γ GLA L is a left ideal if

$$<\mathrm{L},\,\mathrm{N}>\;\subset\;\mathrm{N}. \qquad \ldots(3.8a)$$

A linear subset N of a Γ GLA L is a right ideal if

$$<\mathrm{N},\,\mathrm{L}>\;\subset\;\mathrm{N}. \qquad \ldots(3.8b)$$

An ideal N is Abelian if $<\mathrm{N},\,\mathrm{N}>\;=0$.

An ideal N of L is a Γ graded ideal if

$$\mathrm{N}=\;\bigoplus_{\gamma\in\Gamma}\;N_{\gamma} \qquad \ldots(3.9)$$

Definition 2—The left (right) kernel of a ΓGLA **L** is the set

$$T_l = \{t \in \mathbf{L},\ K(t, l) = 0 \text{ for all } l \in \mathbf{L}\} \qquad \text{...(3.10a)}$$

$$T_n = \{t \in \mathbf{L},\ K(l, t) = 0 \text{ for all } l \in \mathbf{L}\}. \qquad \text{...(3.10b)}$$

For semisimple ΓGLA, the kernels are obviously trivial.

Definition 3—The left (right) orthogonal complement of a subset N of a ΓGLA **L** denoted by $\mathbf{N}_l^{\perp}$ $(\mathbf{N}_t^{\perp})$ is defined as follows

$$\mathbf{N}_l^{\perp} = \{l \in \mathbf{L} : K(l, n) = 0 \text{ for all } n \in \mathbf{N}\} \qquad \text{...(3.11a)}$$

$$\mathbf{N}_r^{\perp} = \{l \in \mathbf{L} :\ K(n, l) = 0 \text{ for all } n \in \mathbf{N}\}. \qquad \text{...(3.11b)}$$

The following lemmas are useful.

Lemma 3—Given $\mathbf{N} \subset \mathbf{L}$, we define

$$\mathbf{N}' = \{n_0 - n_1, \text{ for all } n_0 + n_1 \in \mathbf{N}, \text{ where } n_0 \in \mathbf{L}_0,\ n_1 \in \mathbf{L}_1\}.$$

Then

$$(\mathbf{N}_r')^{\perp} = (\mathbf{N}_l)^{\perp}.$$

PROOF : Let $l \in (\mathbf{N}_r')^{\perp}$ and $l = l_0 + l_1$, $l_0 \in \mathbf{L}_0$, $l_1 \in \mathbf{L}_1$.

Hence $K(n_0 - n_1,\ l_0 + l_1) = 0$. By (2.15), $K(l_0 + l_1, n_0 + n_1) = 0$,

that $l \in (\mathbf{N}_l)^{\perp}$, Hence $(\mathbf{N}_r')^{\perp} \subseteq (N_l)^{\perp}$. The reverse inclusion also holds, hence the result.

When a subset N is Γ graded, $\mathbf{N}' = \mathbf{N}$. Hence for Γ graded ideals, we may define a unique orthogonal complement which is equal to both the left and right orthogonal complements. We finally have :

Lemma 4—$\mathbf{N}^{\perp}$, the orthogonal complement of a Γ graded ideal N of a nondegenerate ΓGLA **L** is a Γ graded ideal.

PROOF : We first show that $\mathbf{N}^{\perp}$ is an ideal of **L**. By Theorem 1c we may write

$$K(\langle l, n^{\perp}\rangle, n) + \epsilon(|l|, |n^{\perp}|)\, K(n^{\perp}, \langle l, n\rangle) = 0 \text{ for all } n \in \mathbf{N},$$
$n^{\perp} \in \mathbf{N}^{\perp}$. $l \in \mathbf{L}$.

The second term vanishes as $\langle l, n\rangle \in \mathbf{N}$ is orthogonal to $\mathbf{N}^{\perp}$. Hence $\langle l, n^{\perp}\rangle$ is orthogonal to N for all $n^{\perp} \in \mathbf{N}^{\perp}$, yielding $\langle l, n^{\perp}\rangle \in \mathbf{N}^{\perp}{}_l$ for all $1 \in \mathbf{L}$, and so $\mathbf{N}^{\perp}$ is is an ideal of **L**. We now show that $\mathbf{N}^{\perp}$ is Γ graded.

Let $n^{\perp} = \sum_{\alpha \in \Gamma} n_{\alpha}^{\perp}$ where $n_{\alpha}^{\perp} \in N_{\alpha}^{\perp}$.

Then as N is Γ graded, if $n = n_{\gamma} \in N_{\gamma}$, $K(n^{\perp}, n) = 0 \Rightarrow$

$$K(\sum_{\alpha \in \Gamma} n_{\alpha}^{\perp}, n_{\gamma}) = 0.$$

But $K(n_{\alpha}^{\perp}, n_{\gamma}) = 0$ for all $\alpha \neq -\gamma$ by (2.14), so $\exists\ \gamma' = -\gamma : n_{\gamma}^{\perp} \in N_{\gamma'}^{\perp}$ and $K(n_{\gamma'}^{\perp}, n_{\gamma}) = 0$. Also $K(n_{\gamma'}^{\perp}, n) = 0$ where $n = \underset{\beta \in \Gamma}{\oplus} n_{\beta}$ because $K(n_{\gamma'}^{\perp}, n_{\beta}) = 0$ for all $\beta \neq -\gamma'$. The above argument holds for all $\gamma \in \Gamma : N = \underset{\gamma \in \Gamma}{\oplus} N_{\gamma}$, hence $N^{\perp} = \underset{\gamma \in \Gamma}{\oplus} N_{\gamma}^{\perp}$. This completes the proof.

Lemma 4 is invaluable in generalising the results of nondegenerate GLA in Ref. 8 to nondegenerate Γ GLA. The results are collected in the following theorem :

Theorem 3—Given a nondegenerate ΓGLA **L** as defined above. Then,

(1) There are no nontrivial Γ graded Abelian ideals.

(2) **L** may be written as a direct sum of Γ graded ideals $\oplus M_{\alpha}$ where each M_{α} is nondegenerate[14] and has no nontrivial Γ graded ideal.

(3) $<\mathbf{L}, \mathbf{L}> = L.$ ...(3.12)

The proofs follow on the same lines as in Mitra and Tripathy[10] and are left as an exercise. We now obtain the last result of this section on the structure of nondegenerate ΓGLA's.

Definition 4—Given a Γ GLA **L** over a Γgraded vector space V. For any $\delta \in \Gamma$, let $D(\mathbf{L}, \epsilon)_{\delta}$ denote the subspace of all elements $D \in gl(V, \in)_{\delta}$, the general linear $\in$ Lie Algebra of v[15] :

$$D(<a, b>) = <D(a), b> + \epsilon(\delta, \alpha) <a, Db>$$

$\forall$ elements $a, b, \in \mathbf{L}$ with degrees $\alpha, \beta \in \Gamma$ respectively, It is easy to see that $D(\mathbf{L}, \epsilon) = \underset{\delta \in \Gamma}{\oplus} D(\mathbf{L}, \epsilon)_{\delta}$ is a Γgraded subalgebra of $gl(\mathbf{L}, \epsilon)$. The elements of $D(\mathbf{L}; \epsilon)_{\delta}$ are called the ϵ derivations of degree δ.

Two Γ graded derivations d, d' of degrees $\delta, \delta' \in \Gamma$ respectively combine by the usual bracket (< >) and

$$<d, d'> = dd' - \epsilon(\delta, \delta') d'd \text{ for all } d, d' \in D. \qquad ...(3.14)$$

We finally have the following theorem.

Theorem 4—All $\in$ derivations of the nondegenerate ΓGLA **L** are inner i. e.

$$D = \mathrm{ad}\mathbf{L}. \qquad ...(3.15)$$

PROOF : We first see that $ad\mathbf{L}$ is a Γ graded ideal of $\mathbf{D}$. We note that for all l, $m \in \mathbf{L}$,

$$<D, \text{ad}l> m = D <l, m> - \in (\delta, | l |)<l, Dm> = <Dl, m> \qquad \ldots(3.16)$$

(using (3.13) and (3.14)).

Now $l \to \text{ad}l$ is a Γ graded homomorphism of $\mathbf{L}$ into $\mathbf{D}$ with kernel, the center of $\mathbf{L}$ which is trivial as $\mathbf{L}$ is non-degenerate. Hence ad$\mathbf{L}$ is a representation of $\mathbf{L}$ in $\mathbf{D}$ so that ad$\mathbf{L}$ has a nondegenerate Killing form K which is also the restriction of the Killing form K' of $\mathbf{D}$ to ad$\mathbf{L}$ as ad$\mathbf{L}$ is an ideal of $\mathbf{D}$[15].

If ad$\mathbf{L}^{\perp}$ be the orthogonal complement of ad$\mathbf{L}$ with respect to K', then the non-degeneracy of K ensure that ad$\mathbf{L} \cap$ ad$\mathbf{L}^{\perp}$ is zero. It also means that $\mathbf{N} = <\text{ad}\mathbf{L}, \text{ad}\mathbf{L}^{\perp}> = 0$ for both ad$\mathbf{L}$ and ad$\mathbf{L}^{\perp}$ are Γ graded ideals and $\mathbf{N}$ is contained in both ad$\mathbf{L}$ and ad$\mathbf{L}^{\perp}$. If $d \in$ ad$\mathbf{L}^{\perp}$, then $0 = <d, \text{ad}l> = \text{ad}dl$ for all $l \in \mathbf{L}$ from which $dl = 0$ for all $l \in \mathbf{L}$, hence $d = 0$. Thus ad$\mathbf{L}^{\perp} = 0$ which immediately implies that ad$\mathbf{L} = D$.

4. COHOMOLOGY OF NONDEGENERATE ΓGLA :

In this section, we generalize the definition of coboundary operator[10] by considering mappings $f : g \otimes g \ldots \otimes g \to V$ of arbitrary degree ρ, $\rho \in \Gamma$, V is a ϕ module of g and $f(X_1, \ldots, X_i, X_{i+1} \ldots X_n) = - \in (| X_i |, | X_{i+1} |) f X_1, \ldots, (X_{i+1}, X_i, \ldots X_n)$ ...(4.1)

for all $X_1, \ldots, X_n \in g$

$| X_i |$ being the degree of $X_i \in g$.

In subsequent discussions, we suppress the subscript ρ in f. Our generalized coboundary operator takes the form

$$\begin{aligned} \delta^n f(X_1, \ldots, X_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} \epsilon(| f |, | X_i |) \,\epsilon\Big(\sum_{k=1}^{i-1} | X_k |, | X_i |\Big) \\ &\times \phi(X_i) f(X_1, \ldots \hat{X}_i \ldots, X_{n+1}) + \sum_{i<j=1} (-1)^{i+j} \epsilon \Big(\sum_{k=1}^{i-1} | X_k |, | X_i |\Big) \\ &\times \epsilon\Big(\sum_{l=1}^{j-1} | X_l |, | X_j |\Big) \epsilon(| X_j |, | X_i |) f(<X_i, X_j>, \ldots, \hat{X}_i \ldots, \\ &\times \hat{X}_j \ldots X_{n+1}) \end{aligned} \qquad \ldots(4.2)$$

The notation $\epsilon\big(\sum_{k=1}^{i-1} | X_k |, | X_i |\big)$ will be shortened to $\epsilon(-i')$

for all elements $X_k \in g$ to which special attention is not being drawn in the text. $\epsilon(| X_j |, | X_i |)$ will be written as $\epsilon(j', i')$. $\hat{X}_i$, means that X_i is to be omitted in $f(X_1, \ldots, X_n)$

The analysis made in the earlier sections allows us to compute the cohomology of a wide class of finite dimensional representations of nondegenerate ΓGLA's. Our results are stated in the following generalisation of Whitehead's lemmas

Theorem 5—Given a nondegenerate ΓGLA g over a commutative field $\mathscr{F}$ of characteristic zero, and ϕ: g → End$V$. Let V be a finite dimensional module of g

(1) If $\phi(C_2)$ is invertible, $H^n_\varphi(\mathrm{g}, V) = 0$, for all $n > 0$;

(2) If $\phi(X)\, v_\beta = 0$ for all $X \in \mathrm{g}$, $v_\beta \in V_\beta$, $\beta \in \Gamma$, then $H^1_\varphi(g, V) = 0$.

Proof : We introduce the notation

$$| X_i | = i', \; | l^\alpha | = \alpha_1, \; | l_\alpha | = \alpha_2.$$

If $f \in Z^n_\varphi$, (g, V), then

$$\delta^n f(X_1, \ldots, X_{n+1}) = 0 \text{ for all } X_1, \ldots X_{n+1} \in \mathrm{g}.$$

(1) $\varphi(C_2\}$ is invertible. Let $X_1 = l^\alpha$, l^α being a member of the dual basis [see eq. (3.1)] of g, Then (4.3) implies (not including $| l^\alpha |$ in $\epsilon(-l')$ of g).

$$\epsilon(|f|, \alpha_1)\, \phi(l^\alpha) f(X_1, \ldots, X_n)$$

$$+ \sum_{i=1}^{n} (-1)^i \epsilon(|f|, i')\, \epsilon(\alpha_1, i')\, \epsilon(-i')\, \phi(X_i)$$

$$\times f(1^\alpha, \ldots, \hat{X}_i, \ldots. X_n)$$

$$+ \sum_{i=1}^{n} (-1)^i \epsilon(-i') f(< l^\alpha, X_i >, \ldots, \hat{X}_i \ldots X_n)$$

$$+ \sum_{i<j=1}^{n} (-1)^{i+j} \epsilon(-l')\, \epsilon(-l')\, \epsilon(j', i')\, \epsilon(\alpha_1, i + j')$$

$$\times f(<X_i, X_j>, l^\alpha, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_n) = 0. \qquad \ldots(4.4)$$

We operate with $\Sigma \in (|f|, \alpha_2)\, K(l_\alpha, l_\beta)\, \phi(l^\beta)$ and obtain, dropping summation symbols α, β over the dummy indices α, β, using (3.3b) and simplifying

$$K(l_\alpha, l_\beta)\, \phi(l^\beta)(l^\alpha) f(X_1, \ldots, X_n)$$

$$+ \sum_{i=1}^{n} (-1)^i \epsilon(-i')\, \epsilon(|f|, i')\, \epsilon(|f|, \alpha_2)\, K(l_\alpha, l_\beta)$$

$$\times \phi(X_i)\, \phi(l^\beta) f(l^\alpha, \ldots, \hat{X}_i, \ldots X_n)$$

$$+ \sum_{i=1}^{n} (-1)^i \epsilon(-i') \epsilon(|f|, i') \epsilon(\alpha_1, i') \epsilon(|f|, \alpha_2)$$

$$\times K(l_\alpha, l_\beta) <\phi(l^\beta) \phi(X_i)> f(l^\alpha, \ldots \hat{X}_i, \ldots X_n)$$

$$+ \sum_{i=1}^{n} (-1)^i \epsilon(-l') \epsilon(|f!, \alpha_2) K(l_\alpha, l_\beta) \phi(l^\beta) f(<l^\alpha, X_i>, \ldots$$

$$\times \hat{X}_i \ldots X_n)$$

$$+ \sum_{i<j=1}^{n} (-1)^{i+j+1} \in(-i') \in(-j') \in(j', i') \in(|f|, \alpha_2)$$

$$\times K(l_\alpha, l_\beta) \phi(l^\beta) f(1^\alpha, <X_i, X_j>, \ldots, \hat{X}_i, \ldots, \hat{X}_j \ \ldots \ X_{n+1})$$

$$= 0. \qquad \ldots(4.5)$$

The third and fourth terms may be written as

$$\sum_{i=1}^{n} (-1)^{i+1} \epsilon(-i') \epsilon(|f!, \alpha_2) \epsilon(+f|, i') K(l_\alpha, l_\beta) <\phi(X_i),$$

$$\times \phi(l^\beta)> f(l^\alpha, \ldots, \hat{X}_i \ldots X_n)$$

$$+ \sum_{i=1}^{n} (-1)^{i+1} \epsilon(-i') \epsilon(|f|, \alpha_2) \epsilon(\alpha_1, i') K(l_\sigma, l_\beta) \varphi(l^\beta)$$

$$\times f(< X_i, l^\alpha >, \ldots, \hat{X}_i, \ldots X_n)$$

$$= \sum_{i=1}^{n} (-1)^{i+1} \epsilon(-i') \epsilon|f|, i') \epsilon(|f|,\alpha_2) K(l_\alpha, l_\beta) L_{i\rho}^{\beta} \phi(l^\rho) f(l^\alpha \ldots, \hat{X}_i \ldots X_n)$$

$$+ \sum_{i=1}^{n} (-)^{i+1} \epsilon(-i'] \epsilon(|f|, \alpha_2) \epsilon(\alpha_1, i') K(l_\alpha, l_\beta) \phi(l^\beta)$$

$$\times L_{i\gamma}^{\alpha} f(l^\gamma, \ldots, \hat{X}_i, \ldots X_n) \qquad \ldots(4.6)$$

where,

$$<X_i, l^\beta> = L_{i\rho}^{\beta} l^\rho \qquad \ldots(4.7a)$$

$$<X_i, l_\beta> = M_{i\beta}^{\rho} l_\rho. \qquad \ldots(47b)$$

Using (3.3a) and noting that $\alpha_2 = \gamma_2 + i'$, $\beta_2 = \rho_2 + i'$,

we get

$$\sum_{i=1}^{n} (-1)^i \epsilon(-i') \epsilon(|f|, i' + \alpha_2) \epsilon(\rho_2 + i', i') \; K(l_\alpha, <X_i, l_\rho>)$$

$$\phi(l^\rho) f(l^\alpha, \ldots, \hat{X}_i \ldots X_n)$$

$$+ \sum_{i=1}^{n} (-1)^j \epsilon(-i') K(<X_i, l_\gamma>, l_\beta) \epsilon(|f|, \gamma_2 + i') \phi(l_\beta) f(l^\gamma,$$

$$\times \ \ldots, \hat{X} \ldots, X_n).$$

Substituting $\alpha_2 \to \gamma_2, \rho_2 \to \beta_2$ in the first term and noting that $\gamma_2 = -(\beta_2 + i')$, we get

$$\sum_{i=1}^{n} (-l)^i \epsilon(-i) \epsilon(|f|, \gamma_2 + i')[K(<X_i, l_\gamma>, l_\beta) + \epsilon(i', \gamma_2) K(l_\gamma,$$

$$<X_i, l_\beta>)] \phi(l^\beta) f(l^\alpha, \ldots, \hat{X}_i, \ldots X_n) \quad \ldots(4.8)$$

which is zero by Theorem (lc).

Putting $\phi(C_2) = K(l_\alpha, l_\beta) \phi(l^\beta) \phi(l^\alpha)$, (4.5) reduces to

$$f(X_1, \ldots, X_n) = \sum_{i=1}^{n} (-1)^{i+1} \epsilon(-i') \epsilon(|f|, i') \phi(X_i) [\epsilon(|f|, \alpha_2)$$

$$\times K(l_\alpha, l_\beta) \phi(C_2^{-1}) \phi(l^\beta) f(l^\alpha, \hat{X}_i, X_n)]$$

$$+ \sum_{i<j=1}^{n} (--1)^{i+j} \epsilon(j', i', i') \epsilon(-i') \epsilon(-j') [\epsilon(|f|, \alpha_2)$$

$$K(l_\alpha, l_\beta) \phi(C_2^{-1}) \phi(l^\beta) f(l^\beta) f(l^\alpha, <X_i, X_j> \ldots \hat{X}_i, \ldots, \hat{X}_j, \ldots X_n)].$$

$$\ldots(4.9)$$

Identifying $w(X_1, \ldots, X_i, \ldots, X_n) = \epsilon(|f|, \alpha_2) K(l_\alpha, l_\beta) \check{\phi}(C_2^{-1}) \phi(l^\beta) f(l^\alpha \ldots \hat{X}_i \ldots X_n).$

...(4.10)

We find that for any $(f(X_1, \ldots, X_n)$ there exists a linear mapping $w : g \otimes \ldots g \to V :$ $f = \delta^{n-1} w$; for all $n > 0$, $(n - 1)$ times

when $\delta^n f = 0$.

This proves the result.

(2) $\phi(X) v_\beta = 0$ for all X $g = > \phi(C_2) v_\beta = 0$.

To prove $H^1_\phi(g, V) = 0$, we see that

$\delta^1 f(X_1, X_2) = 0 \Rightarrow f(<X_1, X_2>) = 0$ for all $X_1, X_2 \in$ g. $f \in Z^1_\phi$ (g, V).

By Theorem 3c,

$$f(X) = 0 = \phi(X)\, v_\beta^- \text{ for all } X \in g,\ v_\beta \in V_\beta,\ \beta \in \Gamma.$$

Hence

$$H^1_\phi \text{ (g, } V) = 0.$$

This completes the proof. The results of our analysis apply to all nondegenerate ΓGLA, e.g. $Z_3 \otimes Z_3$, graded algebras (discussed in Ref. 5) and all Lie superalgebras. In passing, we mention that for the orthosymplectic sequence of semisimple GLA, Whithead's lemmas hold i. e.

$$H^n_\varphi \text{ (g, } V) = 0, (n = 1,2). \qquad ...(4.11)$$

5. Applications of the Cohomology

The two major applications of the cohomology that we discuss below are :

(a) Extensions of.algebras by Abelian ideals.

(b) Reducibility of representations.

(a) *Extensions of Algebras by Abelian Ideals*—This was discussed in sufficient details in Mitra and Tripathy[10]. The only additional remark we wish to make here is that for the class of Γ graded Abelian ideals A which form a representation of semisimple ΓGLA of such that ϕ (C_2) is invertible, the extension of g by A is trivially accomplished by the semidirect sum.

(b) *Reducibility of Representations*—One of the most fundamental theorems in the theory of semisimple Lie Algebras is the theorem of *H.* Weyl which asserts that every finite dimensional representation V of the semisimple Lie Algebra g is semisimple i. e. has the property that every g invariant submodule W of V has a g invariant complement $\overline{W}$, so that it is completely reducible and the study of a representation of a semisimple *LA* reduces to the study of its irreducible representations. In what follows, we formulate the generalised Weyl's reducibility theorem for nondegenerate ΓGLA drawing freely from Jacobson[16] and Vardarajan[17].

Theorem 6—Let g be a ΓGLA over a commutative field $\mathscr{F}$. Then all finite dimensional ϕ modules V of g are semisimple if H^1_σ (g, F) = 0,

where F and σ are defined in eqn. (5.4) and (5.6) respectively.

Proof : Let W be a Γ graded submodule of V invariant under g so that $0 \neq W \neq V$. We select a Γgraded subspace $\overline{W}$ of V complementary to $\overline{W}$. If A is any projection of V onto W i. e.

$$A^2 = A \qquad \ldots(5.1)$$

$$A\,[w_\beta] = w_\beta \text{ for all } w_\beta \in W_\beta \qquad \ldots(5.1\text{b})$$

$$A\,[\bar{w}_\beta] = 0 \text{ for all } \overline{W}_\beta \in W_\beta,\ \beta \in \Gamma. \qquad \ldots(5.1\text{c})$$

It follows trivially that A is Γ homogeneous of degree zero. We now use the following lemma :

Lemma 5—$\overline{W}$ is invariant under ϕ, if and only if

$$<A, \phi\,(X)> = 0 \text{ for all } X \in \mathrm{g}. \qquad \ldots(5.2)$$

PROOF : Let $v_\beta = (w_\beta + \bar{w}_\beta)$ for all $\beta \in \Gamma$.

Then,

$$\forall\ X \in \mathrm{g},\ A\phi\,(X)\,v_\beta = A\phi\,(X)\,(w_\beta + \bar{w}_\beta)$$
$$= \phi\,(X) w_\beta + A\phi\,(X)\,\bar{w}_\beta$$

and

$$\phi\,(X)\,Av_\beta = \phi\,(X)\,w_\beta.$$

Hence

$$<A, \phi\,(X)>\,v_\beta = A\phi\,(X)\,\bar{w}_\beta \qquad \ldots(5.3)$$

Let $\phi\,(X)\,\bar{w}_\beta \in \overline{W}_{|x|+\beta}$, i. e. $\bar{w}$ is invariant under g.

Then the right-hand side of (5.3) is zero, implying that

$$<A, \phi\,(X)> = 0 \text{ for all } \beta \in \Gamma,\ X \in \mathrm{g}.$$

Conversely, from (5.3) using $<A, \phi\,(X) > 0$, we find that

$$\phi\,(X)\,\bar{w}_\alpha \in \overline{W}_{|x|+\alpha} \text{ for all } \alpha,\ |\,X\,| \in \Gamma,\ \bar{w}_\alpha \in \overline{W}_\alpha$$

proving the lemma.

All Γ homogeneous projections B of degree zero of V onto W may not satisfy (5.2), so we must modify the equation in order to be able to construct a Γ homogeneous projection of degree zero A', satisfying the same. To this end, we introduce the vector space F of all endomorphisms of $V : C \in F,\ C\,[V] \subseteq W,\ C\,[W] = \{0\}$. .. (5.4)

Since $0 \neq W \neq V$, it follows that an endomorphism A' of V is a Γ homogeneous projection of degree zero of V onto W if and only if it is of the form $(B - C)$ for a suitable Γ homogeneous $C \in F$ of degree zero such that

$$<\phi\,(X), B> = <\phi\,(X), C> \text{ for all } X \in \mathrm{g}. \qquad \ldots(5.5)$$

It follows from the definition of F that if $D \in F$, then for any endomorphism L of V that leaves W invariant, both LD and DL belong to F. In particular, if we set

$$\sigma(X)D = <\phi(X), D> \text{ for all } X \in g, D \in F. \tag{5.6}$$

Then $\sigma(X) : D \to \sigma(X)D$ is an endomorphism of F for all $X \in g$. We may verify trivially that σ is a representation for

$$\begin{aligned}<\sigma(X), \sigma(Y)> D &= <\phi(X), <\phi(Y)D>> - \epsilon(|X|, |Y|) \\ &\qquad \times <\phi(Y), <\phi(X), D>> \\ &= <<\phi(X), \phi(Y)>, D> = <\phi(<X, Y>), D> \\ &= \sigma(<X, Y>)0\end{aligned}$$

(by the ϵ Jacobi identity).

On the other hand, it follows from the relations

$$Bw_\beta = w_\beta \text{ for all } w_\beta \in W_\beta, \beta \in \Gamma. \qquad \text{...(5.7a)}$$

$$Bv_\beta \in W_\beta \text{ for all } v_\beta \in V_\beta, \qquad \text{...(5.7b)}$$

that for any $X \in g$, $<\phi(X), B>$ is an element of F. Let

$$\theta(X) = <\phi(X), B> \text{ for all } X \in g. \qquad \text{...(5.8)}$$

This θ is a linear map of g in F. We now calculate $d\theta$. We have for all $X, Y \in g$.

$$\begin{aligned}d\theta(X, Y) = {}&< \phi(X), <\theta(Y), B>> - \epsilon(X|, |Y|> <\phi(Y), \\ &<\phi(X), B>> - <\phi(<X, Y\rhd), B> = 0.\end{aligned}$$

Hence

$$\theta \in Z^1_\sigma(g, F), \text{ If } H^1_\sigma(g, F) = 0 \qquad \text{...(5.9)}$$

$$\exists D \in F : \theta(X) = \sigma(X)D \text{ for all } X \in g. \qquad \text{...(5.10)}$$

D being Γ homogeneous of degree zero as may be seen from (5.7) and (5.8).

Thus $<\phi(X), B> = <\phi(X), D>$ for all $X \in g$. which is just (5.5). Hence the conclusion

We now see that all finite dimensional modules of the $Osp\ (1 \mid 2p)$ sequence are reducible as $H^1_\varphi(g, V) = 0$ for all finite dimensional modules of the algebra by (4.11).

We thus reproduce part of the results of the Hochschild Djokovic Theorem[17]. i.e. the $Osp\,(1 \mid 2p)$ sequence is the only sequence of Lie superalgebras for which all finite dimensional modules are completely reducible.

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Appendix I

H^n_ϕ $(g, V) = 0$ $(n = 1, 2)$, for all finite dimensional modules of the orthosympletic $Osp\,(1 \mid 2p)$.

Proof : For this sequence.

$$\phi\,(C_2) = -\sum_{\alpha=1}^{p} \nu_\alpha\,(\nu_\alpha + 2p + 1 - 2\alpha) \qquad \text{...(A1)}$$

ν_α being the eigenvalue of H_α for the highest weight vector[19] H_α being an element of the Cartan subalgebra. From Kac[20], we see that

$$\nu_\alpha = 2p,\; p \in Z^+.$$

It is easy to see that if

$$\phi\,(C_2) = 0,\; \nu_\alpha = 0 \text{ for all } \alpha = 1, \ldots, p$$

hence,

$$\phi\,(X)\,\nu_\beta = 0 \text{ for all } X \in g,\; \nu_\beta \in V_\beta,\; \beta \in \Gamma. \qquad \text{...(A2)}$$

Hence, when Case(l) of Theorm 5 does not apply, Case (2) does, and it follows that H^1_ϕ (g, V) = 0 for all finite dimensional modules V of g.

We now show that H^2_ϕ (g, V) = 0, g being a member of the orthosymplectic sequence and V a finite dimensional module of g.

If $f \in Z^2_\phi$ (g, V), then, $f = f_0 + f_1$, (A3) f_0 and f_1 being defined on V_0 and V_1 respectively.

V_0 is the submodule of V on which (A2) holds and

V_1 the submodule of V on which ϕ (C_2) is invertible.

Both subspaces are invariant under g and have on element in common except 0, also span V, hence $V, = V_0 \oplus V_1$. In considering f_1 : g $\otimes$ g $\rightarrow V_1$, ϕ (C_2) is invertible on V_1 and Theorem 5 applies. In considering f_0 : g $\otimes$ g $\rightarrow V_0$, we see that (A2) applies on V_0. Then if $f_0 \in Z^2_\phi$ (g, V_0),

$$\delta^2 f_0 (X_1, X_2, X_3) = 0$$

$$\Rightarrow f_0 (<X_1, X_2>, X_3) + \epsilon (| X_1 | , | X_2 | + | X_3 |) f_0 (<X_2, X_3>,$$

$$X_1) - \epsilon (| X_2 | , | X_3 |) f_0 (<X_1, X_3>, X_2 = 0. \qquad \text{...(A4)}$$

Let $\mathscr{X}$ denote the Γ graded vector spacs of linear mappings of g into V_0. We make this into a g module by defining for $A \in \mathscr{X}$, $X_1, X_2 \in$ g.

$$\phi (X_1) A (X_2) = - \epsilon (| X_1 | , | A |) A \ (<X_1, X_2>). \qquad \text{...(A5)}$$

This satisfies the module conditions given in Jacobson[16] (Chapter 1) which admit of the following generalization :

(1) $[\phi (X_1) + \phi (X_2)] A (X) = \phi (X_1) A (X) + \phi (X_2) A (X)$

(2) $\alpha\phi (X_1) A (X) = \phi (\alpha X_1) A (X) = \phi (X_1) A (\alpha X)$

(3) $\phi (<X_1, X_2>) A (X) = \phi (X_1) \phi (X_2) A (X) - \epsilon (| X_1 | , | X_2 |)$

$\phi (X_2) \phi (X_1) A \ (X)$

For all $X_1, X_2 \in$ g, $\alpha \in \mathscr{F}$, a commutative field.

For each $X_i \in$ g, we define an element $A_{x_i} \in \mathscr{X}$, as the mapping $A_{x_i} (X_1) = f_0 (X_i, X_1)$ g $\otimes$ g $\rightarrow V_0$.

Then $X_i \rightarrow A_{X_i}$ is a linear mapping of g into $\mathscr{X}$ and

$$A_{<X_2, X_3>} (X_1) = f_0 <X_2, X_3>, X_1) \qquad \text{...(A6)}$$

$$-\epsilon(|A|, |X_3|)\ \epsilon(|X_2|, |X_3|)\ \phi\ (X_3)A_{x_2}(X_1)$$

$$= f_0(X_2, <X_3, X_1>) \qquad \ldots(A7)$$

$$\epsilon(|A|, |X_2|)\ \phi\ (X_2)\ Ax_3(X_1) = -\ \epsilon\ (|X_2|, |X_3|)$$

$$\times\ f_0\ (X_3, <X_2, X_1>). \qquad \ldots(A8)$$

By (A4-A8), A $<X_2,X_3>(X_1) = \epsilon(|A|, |X_2|)\ \phi\ (X_2)\ Ax_3(X_1)$

$$-\ \epsilon(|X_2|, |X_3|)\ \epsilon(|A|, |X_3|)\ \phi\ (X_3)\ Ax_2(X_1).$$

Hence,

$$Ax_1 \in Z^1_\phi\ (\mathrm{g}, x).$$

As H^1_ϕ (g, $\mathscr{X}$) $= 0$ for all finite dimensional modules $\mathscr{X}$ of g, then there exists

$$\rho : \mathrm{g} \to V_0 :$$

$$Ax_i(X_1) = \epsilon(|\rho|, |X_i|)\ \phi\ (X)\ \rho\ (X_1)$$

$$\Rightarrow f(X_i, X_1) = -\ \rho\ (<X_i, X_1>)$$

proving the result for V_0.

Hence H^1_ϕ (g, V) $=$ H^2_ϕ (g, V) $= 0$ for all finite dimensional modules V of the ortho-symplectic sepuence $Osp\,(1\ |\ 2p)$.

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(N, p, q) SUMMABILITY OF JACOBI SERIES

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In this paper, the authors prove a theorem on (N, p, q) summability of Jocobi series at the point $x = 1$.

§1. Let $\{p_n\}$ be a sequence such that $p_0 > 0, p_n > 0, n \geqslant 0, P_n = \sum_{k=0}^{n} p_k \to \infty$ as $n \to \infty$ and

$$\bar{t}_n = P_n^{-1} \sum_{k=0}^{n} P_k S_k \to S \text{ when } n \to \infty$$

then we say that $\{S_n\}$ is summable by Riesz means and we write[3]

$$S_n \to S\ (\bar{N}, p_n) \qquad \ldots(1.1)$$

where $\{S_n\}$ is the sequence of nth partial sums of the series $\sum_{n=0}^{\infty} a_n$. If

$$t_n = P_n^{-1} \sum_{k=0}^{n} p_k S_{n-k} \to S \text{ when } n \to \infty,$$

then we say that $\{S_n\}$ is summable by Nörlund means and we write[3]

$$S_n \to S\ (N, p_n).$$

Let p denotes the sequence $\{p_n\}$, $p_{-1} = 0$. We use similar notations with other letters in place of p. Given two sequences p and q. the convolution $p * q$ is defined by

$$(p * q)_n = \sum_{k=0}^{n} p_{n-k}\, q_k \qquad \ldots(1.3)$$

It is familiar and can be easily verified, that the operation of convolution is commutative and associative and

$$(p * 1)_n = \sum_{k=0}^{n} p_k, \text{ where 1 denotes the sequence } \{1\}.$$

For any sequeece $\{S_n\}$, we write

$$t_n^{p,q} = \frac{1}{(p * q)_n} \sum_{k=0}^{n} p_{n-k}\, q_k\, S_k \qquad \ldots(1.4)$$

Definition—Generalized Nörlund Summability (N, p, q) —If $(p * q)_n \neq 0$ for all n, then the generalized Nörlund transform ((N, p,q) transform) of the sequence $\{S_n\}$ is the sequence $\{t_n^{p,q}\}$. If $t_n^{p,q} \to S$ as $n \to \infty$, then the sequence $\{S_n\}$ is said to be summable by generalized Nörlund method (N, p, q) to S and is denoted by[1]

$$S_n \to S\,(N, p, q). \qquad \ldots(1.5)$$

Two important particular cases of (N, p, q) means are

(i) (N, p_n) mean when $q_n = 1$ for all n

(ii) $(\bar{N}, q_n)$ mean when $p_n = 1$ for all n.

The necessary and sufficient conditions for a (N, p, q) method to be regular are

$$\sum_{k=0}^{n} |\, p_{n-k}\, q_k \,| = O\left(|\,(p * q)_n\,| \right)$$

and

$$p_{n-k} = o\left(|\,(p * q)_n\,| \right),$$

as $n \to \infty$, for every fixed $k \geqslant 0$, for which $q_k \neq 0$.

§2. Let $f(x)$ be defined in the closed interval $[-1, 1]$ such that the function $(1-x)^\alpha (1+x)^\beta f(x) \in L[-1, 1]$, $\alpha > -1$, $\beta > -1$. The Jacobi series corresponding to this function is

$$f(x) \sim \sum_{n=0}^{n} a_n\, P_n^{(\alpha,\beta)}(x) \qquad \ldots(2.1)$$

where

$$a_n = \frac{(2n+\alpha+\beta+1)\,\Gamma(n+1)\,\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}$$

$$\times \int_{-1}^{1} (1-x)^{\alpha} \ (1+x)^{\beta} f(x) \, P_n^{(\alpha,\beta)}(x) \, dx \qquad \ldots(2.2)$$

and $P_n^{(\alpha,\beta)}(x)$ are the well-known Jacobi polynomials.

We write

$$F(\phi) = \{ f(\cos \phi) - A \} \ (\sin \frac{\phi}{2})^{2\alpha+1} (\cos \frac{\phi}{2})^{2\beta+1}. \qquad \ldots(2.3)$$

$[\lambda]$ denotes the integral part of λ and $P(1/\phi) = P_{[1/\phi]}$.

§3. Quite a good amount of work in the ordinary Nörlund summability of Jacobi series at the point $x = 1$ has been done during the last decade. None seems to have so for tackled the problem of (N, p, q) summability of Jacobi series at $x = 1$. In this note we attempt to establish a result on (N, p, q) summability of Jacobi series at $x = 1$, in the form of the following theorem.

Theorem—Let $\{p_n\}$ be a non-negative, non-increasing sequence and $\{q_n\}$ be a non-negative, non-decreasing sequence such that

$$\sum_{k=c}^{n} \frac{P_k}{(\log k) \, k^{\alpha+3/2}} = O\left(\frac{(p * q)_n}{q_n \, n^{\alpha+1/2}} \right) \qquad \ldots(3.1)$$

c is a fixed positive integer,

and

$$q_n \, P_n = O(\log n) \, (p * q)_n). \qquad \ldots(3.2)$$

If

$$F_1(t) = \int_0^t | F(\phi) | \, d\phi = o\,(t^{2\alpha+2}/\log(1/t)) \text{ as } t \to 0 \qquad \ldots(3.3)$$

then the Jacobi series (2.1) is summable (N, p, q) to the sum A, provided that $-\frac{1}{2} \leqslant \alpha < \frac{1}{2}$, $-\frac{1}{2} < \beta$ and the antipole condition

$$\int_{-1}^{b} (1+x)^{(\beta-\alpha-1)/2} \, | f(x) | \, dx < \infty \qquad \ldots(3.4)$$

b fixed, is satisfied.

Note : (i) For $q_n = 1$ for all n, our theorem reduces to the theorem of Gupta[2] which is as follows :

Theorem[2]—Let $\{p_n\}$ be a non-negative, non-increasing sequence such that

$$\sum_{k=a}^{n} \frac{P_k}{k^{\alpha+3/2}\log k} = O\left(\frac{P_n}{n^{\alpha+1/2}}\right), \text{ a being fixed positive integer} \qquad \ldots(3.5)$$

$$\sum_{n=1}^{\infty} \frac{n^{\alpha+1/2}}{P_n} < \infty. \qquad \ldots(3.6)$$

If (3.3) holds good then the series (2.1) is summable (N, p_n) at the point $x = 1$ to the sum A. provided $-\frac{1}{2} \leqslant \alpha, < \frac{1}{2}$, $\beta > -\frac{1}{2}$ and the antipole condition

$$\int_{-1}^{b} (1 + x)^{\beta/2-3/4} \; | f(x) | \, dx < \infty. \qquad \ldots(3.7)$$

b fixed, is satisfied.

It is to be noted that we do not assume (3.6). Also (3.4) is a weaker condition than (3.7), (3.2) is automatically satisfied if $q_n = 1$ for all n.

(ii) (a) If $q_n = 1$ for all n, $p_n = A_n^{\alpha+\delta-1/2}$, $\delta > 0$, $\frac{1}{2} \geqslant \alpha + \delta$

(b) Or, if $p_n = 1$ for all n and sequence $\{q_n\}$ is such that for all sufficiently large n,

$$Q_n \,(= \sum_{k=0}^{n} q_k) = \exp\{(\log n)^2\}, \; n \neq 0.$$

For smaller n we define $\{q_n\}$ to be a non-negative, non-decreasing sequence, all conditions of the theorem are satisfied.

If we take $\{p_n\}$ and $\{q_n\}$ be two sequences satisfying the conditions of the theorem but which are such that

$$(N, p_n) \Rightarrow (N, p, q) \qquad \ldots(3.8)$$

then the theorem does not really give us anything since Gupta's theorem shows that the Jacobi saries is summable (N,p_n) and by (3.8) this gives us that it is summable (N, p, q). It is obvious that (3.8) holds in the example (a).

So we would like to show that it is possible for the conditions of our theorem to be satisfied, but (3.8) to be false. We will now prove that (b) is such an example. Let us write

$$p(z) = \sum_{n=0}^{\infty} p_n z^n, \; c(z) = \frac{1}{p(z)} = \sum_{n=0}^{\infty} c_n z^n.$$

Let $\{t_n\}$ and $u_n\}$ denote respectively the (N, p_n) and (N, p, q) transforms of the sequence $\{S_n\}$, Then

$$u_n = \frac{1}{(p * q)_n} \sum_{k=0}^{n} p_{n-k}\, q_k\, S_k$$

$$= \frac{1}{(p * q)_k} \sum_{k=0}^{n} p_{n-k}\, q_k \sum_{m=0}^{k} c_{k-m}\, P_m\, t_m$$

$$= \frac{1}{(p * q)_n} \sum_{m=0}^{n} P_m\, t_m \sum_{k=m}^{n} p_{n-k}\, c_{k-m}\, q_k$$

$$= \sum_{m=0}^{n} \alpha_{nm}\, t_m, \text{ say.} \qquad \ldots(3.9)$$

In order that (3.8) should hold, it is necessary and sufficient that the transformation (3.9) should be regular. For this, it is necessary that

$$\sum_{m=0}^{n} | \alpha_{nm} | = O\,(1)$$

Or thus, it is necessary that

$$\alpha_{nn} = O\,(1)$$

in other words that

$$P_n\, q_n = O\,((p * q)_n). \qquad \ldots(3.10)$$

Thus it is enough to obtain an example in which the conditions of the theorem are satisfied but (3.10) is fake. For the purpose we use example (b). In example (b) (3.10) is fake, for we have

$$\begin{aligned} P_n\, q_n &= n\, q_n \\ &\neq O\,(Q_n) \\ &= O\,((p * q)_n). \end{aligned}$$

Also (3.2) and (3.1) holds good, for

$$\begin{aligned} P_n\, q_n &= n\, q_n \\ &= O\,(\exp\,(\log n)^2\,(\log n)) \\ &= O\,(Q_n \log n) \\ &= O\,((p * q)_n \log n) \end{aligned}$$

and

$$q_n^{\alpha+1/2} \sum_{k=c}^{n} \frac{P_k}{k^{\alpha+3/2} \log k} = q_n\, n^{\alpha+1/2} \sum_{k=c}^{n} \frac{1}{k^{\alpha+1/2} \log k}$$

$$= O\left(q_n\, n^{\alpha+1/2} \frac{n^{1/2-\alpha}}{\log n} \right)$$

$$= O\left(\frac{nq_n}{\log n} \right) = O\,(Q_n) = O\,((p * q)_n).$$

(iii) An exactly similar theorem may be stated for the point $x = -1$. The usual modifications will have to be made between the parameters α and β.

§4. We require the following lemmas:

Lemmes 1[7] If $\alpha > -1$, $\beta > -1$, then as $n \to \infty$.

$$P_n^{(\alpha,\beta)}(\cos\theta) = O\,(n^{\alpha}),\quad 0 \leqslant \theta \leqslant 1/n \qquad \ldots(4.1)$$

$$= O\,(n^{\beta}),\quad \pi - 1/n \leqslant \theta \leqslant \pi \qquad \ldots(4.2)$$

$$= n^{-1/2}\, k\,(\theta) \left[\cos\,(N\theta + r) + \frac{O\,(1)}{n \sin\theta} \right],$$

$$1/n \leqslant \theta \leqslant \pi - 1/n. \qquad \ldots(4.3)$$

where

$$k\,(\phi) = \pi^{-1/2} \left(\sin\frac{\phi}{2}\right)^{-\alpha-1/2} \left(\cos\frac{\phi}{2}\right)^{-\beta-1/2}$$

$$N = n + \frac{\alpha+\beta+1}{2},\quad r = -\,(\alpha + \tfrac{1}{2})\,\pi/4$$

Lemma 2—The antipole condition (3.4) implies that

$$\int_{\delta}^{\pi} |\,F(\phi)\,| \left(\cos\frac{\phi}{2}\right)^{-\alpha-\beta-1} d\phi < \infty,\ 0 < \delta < \pi. \qquad \ldots(4.4)$$

PROOF: Following Gupta[2] (Or putting $x = \cos\phi$ in (3.4)), we can easily establish the lemma.

Lemma 3—The condition (3.1) implies that

$$q_n\, n^{\alpha+1/2} = o\,((p * q)_n) \qquad \ldots(4.5)$$

PROOF: Following Pandey[5] we can easily establish the lemma.

Lemma 4—If $\{p_n\}$ is a non-negative, non-increasing sequence than for large n, uniformly in $O < \phi \leqslant \pi$, $O \leqslant a \leqslant b \leqslant n$,

$$| \sum_{k=a}^{n} p_k \cos \{(n - k + \rho) \phi - r\} (n - k)^{\alpha+1/2}] = O (n^{\alpha+1/2} P (1/\phi)) \quad \ldots(4.6)$$

where

$$\rho = \frac{\alpha + \beta + 2}{2}, r = (\alpha + \frac{3}{2}) \frac{\pi}{2}, \alpha \geqslant - 1/2.$$

Lemma 5— If $\{p_n\}$ is a non-negative, non-increasing sequence and $\{q_n\}$ is a non-negative non-decreasing sequence then

$$\frac{P_n Q_n}{n+1} \leqslant (p * q)_n \quad \ldots(4.7)$$

PROOF : We have

$$(p * q)_n - P_n Q_n/(n + 1) = \sum_{k=0}^{n} q_k (p_{n-k} - P_n/(n + 1)).$$

The sum of coefficients of q_k (where $k = 0, 1, 2, \ldots$) on the R. H. S. of the above equation is 0. For fixed n, the coefficients are non-decreasing with k. So for a given n, the coefficients must be all $\leqslant 0$ upto a certain point and all positive beyond that point. Let k_0 be the greatest value of k for which the cyefficient is $\leqslant O$, where k_0 depends on n. If $k \leqslant k_0$ then

$$p_{n-k} - \frac{P_n}{n + 1} \leqslant 0$$

and

$q_k \leqslant q_{k_0}$ by difinition of $\{q_n\}$.

Hence

$$q_k \left(p_{n-k} - \frac{P_n}{n + 1}\right) \geqslant q_{k_0} \left(p_{n-k} - \frac{P_n}{n + 1}\right).$$

If $k > k_0$, we have

$$p_{n-k} - \frac{P_n}{n + 1} > 0.$$

and

$$q_k > q_{k_0}.$$

Hence, once again we have

$$q_k \left(p_{n-k} - \frac{P_n}{n + 1}\right) \geqslant q_{k_0} \left(p_{n-k} - \frac{P_n}{n + 1}\right).$$

Therefore

$$(p * q)_n - \frac{P_n Q_n}{n+1} = \sum_{k=0}^{n} q_k \left(p_{n-k} - \frac{P_n}{n+1} \right)$$

$$\geqslant q_{k_0} \sum_{k=0}^{n} \left(p_{n-k} - \frac{P_n}{n+1} \right)$$

$$= 0$$

which proves the result.

Lemma 6—Under the hypothesis of theorem

$$\sum_{\nu=0}^{n} q_\nu \, \nu^{\alpha-1/2} = O\,(q_n \, n^{\alpha-1/2}) \qquad \ldots(4.8)$$

PROOP : We have

$$Q_n \, n^{\alpha-1/2} - Q_{n-1}\,(n-1)^{\alpha-1/2} = q_n \, n^{\alpha-1/2} - Q_{n-1}\,\{(n-1)^{\alpha-1/2} - n^{\alpha-1/2}\}.$$

But since $\{q_n\}$ is non-decreasing

$$Q_{n-1} \leqslant Q_n \leqslant (n+1)\, q_n.$$

Also

$$(n-1)^{\alpha-1/2} - n^{\alpha-1/2} \sim (\tfrac{1}{2} - \alpha)\, n^{\alpha-3/2}.$$

Let d be any constant chosen so that $\frac{1}{2} - \alpha < d < 1$, then for sufficiently large n, say for $n \geqslant n_0$, we have $Q_{n-1}\,\{(n-1)^{\alpha-1/2} - n^{\alpha-1/2}\} \leqslant d \, q_n \, n^{\alpha-1/2}$. Therefore,

$$Q_n \, n^{\alpha-1/2} - Q_{n-1}\,(n-1)^{\alpha-1/2} \geqslant (1-d)\, q_n \, n^{\alpha-1/2}$$

Hence, summing we get

$$\sum_{\nu=n_0}^{n} q_\nu \, \nu^{\alpha-3/2} \leqslant \frac{1}{1-d} \{Q_n \, n^{\alpha-1/2} - Q_{n_0} - 1\,(n_0 - 1)^{\alpha-1/2}\}$$

$$\leqslant \frac{1}{1-d} Q_n \, n^{\alpha-1/2}. \qquad \ldots(a)$$

But

$$\sum_{\nu=1}^{n_0-1} q_\nu \, \nu^{\alpha-1/2} = O\,(Q_n \, n^{\alpha-1/2}) \qquad \ldots(b)$$

since the L. H. S. is a constant and $Q_n \, n^{\alpha-1/2} \to \infty$ as $n \to \infty$, adding (a) and (b) we are done.

Lemma 7—Under the hypothesis of the theorem

$$\sum_{k=0}^{n-1} p_k\, q_{n-k}\, (n-k)^{\alpha-1/2} = O\,((p * q)_n\, n^{\alpha-1/2}). \qquad ...(4.9)$$

PROOF : Let M denote a constant, possibly different at each occurrence. For $k \leqslant n/2$.

$$(n-k)^{\alpha-1/2} \leqslant (\tfrac{1}{2})^{\alpha-1/2}\, n^{\alpha-1/2} = Mn^{\alpha-1/2}.$$

So the contribution to the sum on the left of (4.9) of the range $0 \leqslant k \leqslant n/2$ is

$$\leqslant M\, n^{\alpha-1/2} \sum_{0\leqslant k\leqslant n/2} p_k\, q_{n-k}$$

$$\leqslant M\, n^{\alpha-1/2} \sum_{k=0}^{n} p_k\, q_{n-k}$$

$$= M\, n^{\alpha-1/2}\, (p * q)_n.$$

For the part of the sum with $k > n/2$, we have

$$p_k \leqslant \frac{P_k}{k+1} \leqslant \frac{P_n}{\frac{1}{2}(n+1)} = M \frac{P_n}{n+1}.$$

Thus the contribution of this part is

$$\leqslant M \frac{P_n}{n+1} \sum_{n/2<k\leqslant n-1} q_{n-k}\, (n-k)^{\alpha-1/2}.$$

Now let m be the greatest integer with $n - m > n/2$. Then the sum is equal to

$$M \frac{P_n}{n+1} \sum_{\nu=1}^{m} q_\nu\, \nu^{\alpha-1/2} \leqslant M \frac{P_n}{n+1}\, Q_m\, m^{\alpha-1/2} \qquad \text{(by Lemma 6)}$$

$$\leqslant M \frac{P_n\, Q_n}{n+1}\, n^{\alpha-1/2}$$

$$\leqslant M\, n^{\alpha-1/2}\, (p * q)_n \qquad \text{(by Lemma 5)}$$

which proves the result.

Lemma 8—Let

$$N(\phi) = \frac{2^{\alpha+\beta+1}}{(p * q)_n} \sum_{k=0}^{n-1} p_k\, q_{n-k}\, \lambda_{n-k}\, P_{n-k}^{(\alpha+1,\beta)}\, (\cos\phi)$$

where

$$\lambda_n = \frac{2^{-\alpha-\beta-1}\,\Gamma(n+\alpha+\beta=2)}{\Gamma(\alpha+1)\,\Gamma(n+\beta+1)} \cong \frac{2^{-\alpha-\beta-1}}{\Gamma(\alpha+1)}\, n^{\alpha+1}$$

then, for $\frac{1}{2} > \alpha \geqslant -\frac{1}{2}$, $\beta > -\frac{1}{2}$, and if $\{p_n\}, \{q_n\}$ satisfies the hypothesis of the theorem.

$$N(\phi) = O(n^{2\alpha+2}) \text{ if } 0 \leqslant \phi \leqslant 1/n \qquad \ldots(4.10)$$

$$= O(n^{\alpha+\beta+1}) \text{ if } \pi - 1/n \leqslant \phi \leqslant \pi \qquad \ldots(4.11)$$

$$= O\left[\frac{q_n\, n^{\alpha+1/2}\, P(1/\phi)}{(p * q)_n}\left(\sin\frac{\phi}{2}\right)^{-\alpha-3/2}\left(\cos\frac{\phi}{2}\right)^{-\beta-1/2}\right]$$

$$+\, O\left[n^{\alpha-1/2}\left(\sin\frac{\phi}{2}\right)^{-\alpha-5/2}\left(\cos\frac{\phi}{2}\right)^{-\beta-3/2}\right],$$

$$\frac{1}{n} \leqslant \phi \leqslant \pi - 1/n. \qquad \ldots(4.12)$$

PROOF : For $0 \leqslant \phi \leqslant 1/n$, using (4.1) we get

$$N(\phi) = O\left[(p * q)_n^{-1} \sum_{k=0}^{n-1} p_k\, q_{n-k}\,(n-k)^{2\alpha+2}\right]$$

$$= O(n^{2\alpha+2})$$

If $\pi - 1/n \leqslant \phi \leqslant \pi$, using (4.2) we get (4.11).

If $1/n \leqslant \phi \leqslant \pi - 1/n$, we have, with notation as in Lemma 4.

$$N(\phi) = \frac{O(1)}{(p * q)_n} \sum_{k=0}^{n-1} p_k\, q_{n-k}\,(n-k)^{\alpha+1/2}\left(\sin\frac{\phi}{2}\right)^{-\alpha-3/2}$$

$$\left(\cos\frac{\phi}{2}\right)^{-\beta-1/2}\left[\cos\{(n-k)\phi + \rho\phi - r\} + \frac{O(1)}{(n-k)\sin\phi}\right]$$

Since, for fixed n, q_{n-R} is non-increasing, we can deal with the first term on the right by first using the second mean value theorem and then applying Lemma 4.

To deal with second term on the right we apply the result of Lemma 7, and this proves (4.12).

§5. Proof of the theorem : Folling Obreckkoff[4] the nth partial sum of the series (2.1) at the point $x = 1$ is given by

$$S_n(1) = 2^{\alpha+\beta+1}\int_0^{\pi}\left(\sin\frac{\phi}{2}\right)^{2\alpha+1}\left(\cos\frac{\phi}{2}\right)^{2\beta+1} f(\cos\phi)\, S_n'(1, \cos\phi)$$

$$d\phi$$

where $S_n'(1, \cos\phi)$ denotes the n-th partial sum of the series

$$\sum_m \frac{P_m^{(\alpha,\beta)}(1)\, P_m^{(\alpha,\beta)}(\cos\phi)}{g_m}$$

where

$$g_n = \frac{(2n+\alpha+\beta+1)\,\Gamma(n+1)\,\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}.$$

Rao[6] has shown that

$$S_n'(1, \cos\phi) = \lambda_n P_n^{(\alpha+1,\beta)}(\cos\phi).$$

Therefore

$$S_n(1) - A = 2^{\alpha+\beta+1}\lambda_n \int_0^\pi \left(\sin\frac{\phi}{2}\right)^{2\alpha+1}\left(\cos\frac{\phi}{2}\right)^{2\beta+1}\{f(\cos\phi) - A\}$$

$$P_n^{(\alpha+1,\beta)}(\cos\phi)\, d\phi$$

$$= 2^{\alpha+\beta+1}\lambda_n \int_0^\pi F(\phi)\, P_n^{(\alpha+1,\beta)}(\cos\phi)\, d\phi$$

where λ_n is defined as in Lemma 8.

The (N, p, q) means of the series (2.1) at $x = 1$ is given by

$$t_n^{p,q} = \frac{1}{(p * q)_n}\sum_{k=0}^n p_k\, q_{n-k}\, S_{n-k}(1)$$

or

$$t_n^{p,q} - A = \frac{1}{(p * q)_n}\sum_{k=0}^n p_k\, q_{n-k}\{S_{n-k}(1) - A\}$$

$$= \int_0^\pi F(\phi)\, N(\phi)\, d\phi + \frac{p_n\, q_0\, 0(1)}{(p * q)_n}\int_0^\pi F(\phi)\, d\phi$$

Since $\int_0^\pi F(\phi)\, d\phi$ is a finite constant, by assumption, second term on the right is $o(1)$ an $n \to \infty$. Hence in order to prove theorem we have to show that

$$I = \int_0^\pi F(\phi)\, N(\phi)\, d\phi = o(1) \text{ as } n \to \infty.$$

Let us write

$$I = [\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi-1/n} + \int_{\pi-1/n}^{\pi}]d\phi$$

$$= I_1 + I_2 + I_3 + I_4, \text{ say.}$$

where δ is a suitable chosen constant. Now

$$I_1 = \int_0^{1/n} | F(\phi) \, O(n^{2\alpha+2}) \, d\phi \text{ from (4.10)}$$

$$= O \; (n^{2\alpha+2}) \, o \left(\frac{n^{-2\alpha-2}}{\log n} \right)$$

$$= o\,(1), \text{ as } n \to \infty.$$

Coming to I_2, we have

$$I_2 = O \left(\frac{q_n \, n^{\alpha+1/2}}{(p * q)_n} \right) \int_{1/n}^{\delta} \frac{| F(\phi) | \;\; P\,(1/\phi)}{\phi^{(2\alpha+3)/2}} \, d\phi$$

$$+ \; O \;\; (n^{\alpha-1/2}) \int_{1/n}^{\delta} \frac{| F\,(\phi) \,|}{\phi^{(2\alpha+5)/2}} \, d\phi.$$

$$= I_{2,1} + I_{2,2} \text{ say}$$

Given $\epsilon > 0$, let δ be chosen so that

$$| F_1\,(\phi) \,| \; \leqslant \; \frac{\epsilon\phi^{2\alpha+2}}{\log\,(1/\phi)} \cdot O < \phi \leqslant \delta.$$

Then

$$| \, I_{2,1}\,(\phi) \,| \leqslant \frac{M \; q_n \; n^{\alpha+1/2}}{(p \, * \, q)_n} \int_{1/n}^{\delta} \frac{| \; F\,(\phi) \; P_{[\,1/\phi]}}{\phi^{(2\alpha+3)\,/2}} \; d\phi$$

where M is a positive constant, which may be different at each occurrence. Hence

$$| \, I_{2,1} \, | \;\; \leqslant \;\; \frac{M \;\; q_n \; n^{\alpha+1/2}}{(p \, * \, q)_n} \left\{ \left[\; \frac{F_1\,(\phi) \; P_{[\,1/\phi]}}{\phi^{(2\alpha+3)\,/2}} \; \right]_{1/n}^{\delta} - \int_{1/n}^{\delta} F_1\,(\phi) \right.$$

$$\left. \times \, d \left(\; \frac{P_{[\,1/\phi]}}{\phi^{(2\alpha+3)\,/2}} \; \right) \right\}$$

$$= I_{2,1,1} + I_{2,1,2} \text{ say.}$$

If $M(\delta)$ denotes a constant depending on δ, we see that, for fixed δ.

$$I_{2,1,1} = \frac{M(\delta)\, q_n\, n^{\alpha+1/2}}{(p * q)_n} + o\left(\frac{q_n\, P_n}{(p * q)_n \log n}\right)$$

$$= o(1) \text{ (by Lemma 3 and (3.2))}$$

and

$$I_{2,1,2} \leqslant \frac{M \epsilon\, q_n{}^{\alpha+1/2}}{(p * q)_n} \int_{1/n}^{\delta} \frac{\phi^{2\alpha+2}}{\log(1/\alpha)} \left| d\left(\frac{P_{[1/\phi]}}{\phi^{(2\alpha+3)/2}}\right)\right|$$

$$= \frac{M \epsilon\, q_n\, n^{\alpha+1/2}}{(p * q)_n} \int_{1/\delta}^{n} \frac{x^{-2\alpha-2}}{\log x}\, d\{P_{[x]}\; x^{(2\alpha+3)/2}\}$$

$$= \frac{M \epsilon\, q_n\, n^{\alpha+1/2}}{(p * q)_n} \int_{1/\delta}^{n} \frac{x^{-2\alpha-2}}{\log x} \{x^{(2\alpha+3)/2}\, d\, P_{x}]$$

$$+ (2\alpha + 3)/2 \cdot x^{(2\alpha+1)/2}\; P_{[x]}\, dx]$$

$$= \frac{M \epsilon\, q_n\, n^{\alpha+1/2}}{(p * q)_n} \left[\int_{1/\delta}^{n} \frac{x^{-(2\alpha+1)/2}}{\log x}\, d\, P_{[x]}\right.$$

$$\left. + (2\alpha + 3)/2 \,.\, \int_{1/\delta}^{n} \frac{x^{-(2\alpha+3)/2}}{\log x}\, P_{[x]}\, dx\right]$$

$$= \frac{M \epsilon\, q_n\, n^{\alpha+1/2}}{(p * q)_n} [J + (2\alpha + 3)/2K], \text{ say.}$$

Since $P_{(x)}$ has a jump of p_k at $x = k$ (and is elsewhere constant).

$$J = \sum_{k=0}^{n} \frac{p_k}{k^{(2\alpha+1)/2} \log k} \quad \text{where } c \text{ is a fixed positive constant}$$

$$= O\left(\sum_{k=0}^{n} \frac{P_k}{k^{(2\alpha+3)/2} \log k}\right)$$

Also

$$K \leqslant \sum_{k=c-1}^{n-1} P_k \int_{k}^{k+1} \frac{x^{-(2\alpha+3)/2}}{\log x}\, dx$$

$$= O\left\{\sum_{k=\varepsilon-1}^{n-1} \frac{P_k}{k^{(2\alpha+3)/2}\log k}\right\}.$$

Hence

$$| I_{2,1,3} | \leqslant M\,\epsilon. \text{ from (3.1).}$$

Now

$$| I_{2,2} | \leqslant M\, n^{\alpha-1/2} \int_{1/n}^{\delta} | F(\phi) |\, \phi^{-(2\alpha+5)/2}\, d\phi$$

$$= n^{\alpha-1/2} \left\{ M \left[F_1(\phi)\, \phi^{-(2\alpha+5)/2} \right]_{1/n}^{\delta} + M \int_{1/n}^{\delta} F_1(\phi)\, \phi^{-(2\alpha+7)/2}\, d\phi \right\}$$

(The two M's may be different)

$$= I_{2,2,1} + I_{2,2,2}, \text{ say.}$$

Therefore

$$I_{2,2,1} = M(\delta)\, n^{\alpha-1/2} + o(1)$$

$$= o(1), \text{ as } n \to \infty$$

and

$$| I_{2,2,2} | \leqslant M\,\epsilon\, n^{\alpha-1/2} \int_{1/n}^{\delta} \frac{\phi^{\alpha-3/2}}{(\log(1/\phi))}\, d\phi$$

$$= M\epsilon\, n^{\alpha-1/2} \int_{1/\delta}^{n} \frac{x^{-(2\alpha+1)/2}}{\log x}\, dx$$

$$\leqslant M\,\epsilon \text{ because } \alpha < 1/2.$$

Thus

$\lim\limits_{n\to\infty}$ Sup $| I_2 |$ can be made arbitrarily small by choice of δ and thus it is enough to prove that, having fixed δ, we have $I_3 \to 0$, $I_4 \to 0$, as $n \to \infty$. Take, then, δ as fixed. Then

$$I_3 = O\left(\frac{q_n n^{\alpha+1/2}}{(p * q)_n} \right) \int_{\delta}^{\pi-1/n} | F(\phi) | \left(\sin \frac{\phi}{2}\right)^{-\alpha-3/2} \left(\cos \frac{\phi}{2}\right)^{-\beta-1/2}$$

$$\times\, P_{[1/\phi]}\, d\phi$$

$$+ \; O\,(n^{\alpha-1/2}) \int_{\delta}^{\pi-1/n} | \, F(\phi) \, | \left(\sin \; \frac{\phi}{2}\right)^{-\alpha-5/2} \left(\cos \; \frac{\phi}{2}\right)^{-\beta-3/2} d\phi$$

$$= I_{3,1} + I_{3,2}, \text{ say.}$$

Since $(\sin \dfrac{\phi}{2})^{-\alpha-3/2}$ is bounded for $\delta \leqslant \phi \leqslant \pi$ and since $P[_{1/\phi}]$ is bounded and $-\beta -\frac{1}{2} > -\beta - \alpha - 1$. We have

$$I_{3,1} = O\left(\frac{q_n \, n^{\alpha+1/2}}{(p * q)_n}\right) \int_{\delta}^{\pi-1/n} | \, F(\phi) \, | \left(\cos \; \frac{\phi}{2}\right)^{-\alpha-\beta-1} d\phi$$

$$= O\left(\frac{q_n \, n^{\alpha+1/2}}{(p * q)_n}\right) \text{ by} \qquad \ldots(4.4)$$

$$= o\,(1) \text{ as } n \to \infty. \text{ by} \qquad \ldots(4.5)$$

We divide $I_{3,2}$ into $\int_{\delta}^{\delta'}$ and $\int_{\delta'}^{\pi-1/n}$

Given any $\epsilon' > 0$ we can choose δ' so that

$$\int_{\delta'}^{\pi} \left(\cos \; \frac{\phi}{3}\right)^{-\alpha-\beta-1} | \, F(\phi) \, | \, d\phi \leqslant \epsilon'.$$

The contribution to $I_{3,\,2}$ of the range $(\delta', \pi - 1/n)$ is less than or equal a to constant times

$$n^{\alpha-1/2} \int_{\delta'}^{\pi-1/n} | \, F(\phi) \, | \left(\cos \; \frac{\phi}{2}\right)^{-\beta-3/2} d\phi$$

$$= n^{\alpha-1/2} \int_{\delta'}^{\pi-1/n} | \, F(\phi) \, | \left(\cos \frac{\phi}{2}\right)^{-\beta-\alpha-1} \left(\cos \; \frac{\phi}{2}\right)^{\alpha-1/2} d\phi$$

$$\leqslant M\,\epsilon'.$$

Since in the range considered $\left(\cos \; \dfrac{\phi}{2}\right)^{\alpha-1/2}$ is $O\!\left(\left(\dfrac{1}{n}\right)^{\alpha-1/2}\right)$. Thus the lim sup of the contribution of this range can be made arbitrarily small by choice of ϵ'. So that it is enough to prove that for fixed δ', the contribution of the range (δ', δ') tends to 0. But for fixed δ',

$$\int_{\delta}^{\delta'} | F(\phi) | \left(\sin \frac{\phi}{2}\right)^{-\alpha-5/2} \left(\cos \frac{\phi}{2}\right)^{-\beta-3/2} d\phi$$

is a constant, so the contribution is

$$0 (n^{\alpha-1/2}) \to 0, \ \alpha < 1/2.$$

Finally

$$I_4 = 0 (n^{\alpha+\beta+1}) \int_{\pi-1/n}^{\pi} | F(\phi) | d\phi.$$

But $n^{\alpha+\beta+1} = 0 \left(\left(\cos \frac{\phi}{2}\right)^{-\alpha-\beta-1} \right)$ uniformly in $\pi - 1/n \leqslant \phi \leqslant \pi$, whence it follows at once that $I_4 \to 0$.

This completes the proof of the theorem.

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APPROXIMATION OF A FUNCTION BY THE $F(a, q)$ TRANSFORM OF ITS FOURIER SERIES

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Results on the order of approximation of a 2π-periodic continuous function by the Euler or Taylor means of the sequence of partial sums of its Fourier series are extended to general class of $F(a, q)$ transform of which these and other transforms known as Kreisverfahran are special cases.

1. INTRODUCTION

Let $C[0, 2\pi]$ denote the class of all continuous 2π periodic functions. If $f \in C[0, 2\pi]$, ω_f denotes its modulus of continuity. Let the Fourier series associated with $f \in C[0, 2\pi]$ at x be

$$a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \qquad \text{...(1.1)}$$

As usual let us write

$$\phi_x(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}. \qquad \text{...(1.2)}$$

The kth partial sum $s_k(x)$ of the Fourier series is given by

$$s_k(x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \frac{\phi_x(t)}{\sin \frac{1}{2}t} \sin(k + \tfrac{1}{2})\, t\, dt. \qquad \text{...(1.3)}$$

The family $F(a, q)$ of summability methods was introduced by Meir[6]. An $F(a, q)$-transform of the sequence $\{s_k(x)\}$, the sequence of partial sums of the Fourier series of $f \in C[0, 2\pi]$ at x, is

$$\sigma_p(x) = \sigma_p(a, q, f, x) = \sum_{k=0}^{\infty} c_k(p)\, s_k(x),\ c_k(p) \geqslant 0 \qquad \text{...(1.4)}$$

where, for $g(q, k)$ defined by

$$g(q, k) = \sqrt{a/\pi q}\, \exp\{-aq^{-1}(k-q)^2\},\ a > 0,\ q = q(p) \qquad \text{...(1.5)}$$

which is a positive non-decreasing function of a continuous or discrete parameter p, which tends to infinity as $p \to \infty$, and for some fixed γ, $\frac{1}{2} < \gamma < \frac{2}{3}$,

$$c_k(p) = q(q, k)\left\{1 + O\left(\frac{|k-q|+1}{q}\right) + O\left(\frac{|k-q|^3}{q^2}\right)\right\} \qquad \ldots(1.6)$$

as $p \to \infty$ uniformly in k for $|k - q| \leqslant q^{\gamma}$, while

$$\sum_{|k-q|>q^{\gamma}} (k+1)\, c_k(p) = O\{\exp(-q^{\mu})\} \qquad \ldots(1.7)$$

as $p \to \infty$, for some positive number μ independent of p.

The family $F(a, q)$ is known[5] to contain the summability methods of generalised Borel, Euler, Taylor, S_β (defined explicitly later) and Valiron. It is known[5] that

$$\sum_{k=0}^{\infty} c_k(p) = 1 + O(q^{-1/2}). \qquad \ldots(1.8)$$

The summability methods of Euler, Taylor, S_β and Borel satisfy (1,8) in the stronger form

$$\sum_{k=0}^{\infty} c_k(p) = 1. \qquad \ldots(1.9)$$

In what follows ω is a positive non-decreasing function such that

$$\omega(t)/t^{1/2} \text{ is non-increasing function of } t \in [0, \pi]. \qquad \ldots(1.10)$$

As a consequence of this condition (1.10) on ω, for $\lambda > 1$, we get

$$\omega(\lambda t) \leqslant \gamma\lambda\, \omega(t)\ (t > 0). \qquad \ldots(1.11)$$

In the sequel in order relations involving q it is to be understood that $p \to \infty$.

Chui and Holland[1] proved that the order of approximation of functions in the class Lip α by either Euler-$(E, 1)$ means or Taylor means of Fourier series can be re-reduced to Jackson order provided, in each case, a suitable integrability condition is imposed upon $\phi_x(t)$. Xie[8] extended this result to Euler-(E, q) ($q > 0$ means in the context of continuous 2π periodic functions Chui *et al*[3] (p.373, Corollary 5.15) seem to have obtained the analogue of the above result for the Borel method, but the details do not seem to have been published. We append the details pertaining to the analogue for the S_β method in Thsorem 2. We extend this result to the family $F(a, q)$ in the context of the class Lip α, but with the restriction $0 < \alpha \leqslant \frac{1}{2}$.

We prove the following results

Theorem 1—Let n be the integral part of $q = q(p)$. Set $m = n + 1$. Let $f \in C[0, 2\pi]$ such that $\omega_f \geqslant \omega$, and satisfy the condition

$$\int_{u(m)}^{v(m)} \frac{| \phi_x(t) - \phi_x(t + u(m)) |}{t} \exp(-mt^2/4a)\, dt = O\left(\omega\left(\frac{1}{m}\right)\right) \qquad (1.12)$$

uniformly in x. Then

$$\max_{0 \leqslant x \leqslant 2\pi} | \sigma_p(x) - f(x) | = O\left(\omega\left(\frac{1}{m}\right)\right) \qquad ...(1.13)$$

where

$$u(m) = \pi/m + \tfrac{1}{2},\ v(m) = [u(m)]^{\eta},\ 0 < \eta < \tfrac{1}{2}. \qquad ...(1.14)$$

In view of (1.10), Theorem 1 immediately yields the

Corollary--If $\omega(t) = t^{\alpha}$, $0 < \alpha \leqslant \frac{1}{2}$ and $f \in C[0, 2\pi]$ such that $\omega_f \leqslant \omega$ and satisfy the condition (1.12) uniformly in x, then

$$\max_{0 \leqslant x \leqslant 2\pi} | \sigma_p(x) - f(x) | = O(m^{-\alpha}). \qquad ...(1.15)$$

This corollary with the restriction on α weakened to $0 < \alpha < 1$, is due to Chui and Holland[1] for the methods of Euler-$(E, 1)$ and Taylor and due to Xie[8] for the method of Euler (E, q) $(q > 0)$. The following Theorem 2 is the analogue of these results for the method S_β $(0 < \beta < 1)$ for which the transform $\sigma_p \equiv S_\beta^p$ of the sequence $\{s_k\}$ is defined with

$$c_k(p) = (1 - \beta)^{p+1} \binom{p+k}{k} \beta^k,\ k, p = 0, 1, 2, \ldots.$$

Theorem 2 – Let $f \in \mathrm{Lip}\,\alpha$, $0 < \alpha < 1$. Let S_β^p $(p = 0, 1, 2, \ldots)$ denote the pth S_β-mean of the Fourier series.

If

$$\int_{a(p)}^{b(p)} \frac{| \phi_x(t) - \phi_x(t + a(p)) |}{t} \exp\{-\tfrac{1}{2} p\beta (t/1-\beta)^2\}\, dt$$

$$= O(p^{-\alpha}), \qquad ...(1.16)$$

where

$$a(p) = \frac{\pi(1 - \beta)}{(p + 1)\beta},\ b(p) = [a(p)]^{\delta},\ \frac{1 + \alpha}{3 + \alpha} < \delta < \tfrac{1}{2}, \text{ then}$$

$$\max_{0 \leqslant x \leqslant 2\pi} | S_\beta^p(x) - f(x) | = O(p^{-\alpha}). \qquad ...(1.17)$$

2. Preliminary Results

To prove the results in section 1 we need the following lemmas.

Lemma 1—If $q = q(p)$ is an integer valued function of p, then, for $\frac{1}{2} < \gamma < \frac{2}{3}$ we have

$$\int_0^\pi \frac{\phi_x(t)}{\sin t/2} \sum_{|k-q| \leqslant q^\gamma} g(q, k) \sin(k + \tfrac{1}{2})\, t\, dt$$

$$= \int_0^\pi \frac{\phi_x(t)}{\sin t/2} \exp(-\, qt^2/4a) \sin(q + \tfrac{1}{2})\, t\, dt$$

$$+ O(q^{3\gamma/2}.\ \exp(-\, aq^{2\gamma-1})). \qquad \ldots(2.1)$$

Proof : Following the proof of Lemma 3.2 of Ikeno[4] we have

$$\sum_{|k-q| \leqslant q^\gamma} g(q, k) \sin(k + \tfrac{1}{2})\, t\, dt$$

$$= \exp(-\, qt^2/4a) \sin(q + \tfrac{1}{2})\, t$$

$$+ O\{ \sum_{|r> | q^\gamma} \sqrt{a/\pi q} \exp(-\, ar^2/q) \mid \sin(r + q + \tfrac{1}{2})t| \},$$

where $r = k - q$. Now we estimate the error term :

$$\sum_{|r| > q^\gamma} \sqrt{a/\pi q} \exp(-ar^2/q) \mid \sin(r + q + \tfrac{1}{2})t \mid$$

$$\leqslant \sum_{|r| > q^\gamma} \sqrt{a/\pi q} \exp(-ar^2/q)(\,|r| + q + \tfrac{1}{2})\, t$$

$$= 2\sqrt{a/\pi q}\, t \{ \sum_{r>q^\gamma} r \exp(-ar^2/q)$$

$$+ (q + \tfrac{1}{2}) \sum_{r>q^\gamma} \exp(-\, ar^2/q)\}.$$

Now, for large q,

$$\sum_{r>q^\gamma} r \exp(-ar^2/q) \leqslant q^\gamma \exp(-aq^{2\gamma-1}) + \int_{q^\gamma}^\infty y \exp(-ay^2/q)\, dy$$

$$= q^\gamma \exp(-aq^{2\gamma-1}\} + \frac{q}{2a} \int_{aq^{2\gamma-1}}^\infty e^{-z}\, dz.$$

Using the fact that, for real θ

$$\int_\lambda^\infty z^\theta e^{-z}\, dz = O(\lambda^\theta c^{-\lambda}), \text{ as } \lambda \to \infty \qquad \ldots(2.2)$$

we get

$$\sum_{r>q^\gamma} r \exp(-ar^2/q) \leqslant q^\gamma \exp(-aq^{2\gamma-1}) + O(q \exp(-aq^{2\gamma-1})).$$

Similarly, we can prove

$$\sum_{r>q^\gamma} \exp(-ar^2/q) \leqslant \exp(-aq^{2\gamma-1}) + O(q^{1-\gamma} \exp(-aq^{2\gamma-1})).$$

Thus

$$\sum_{|r|>q^\gamma} \sqrt{a/\pi q} \exp(-ar^2/q) \mid \sin(r+q+\tfrac{1}{2}) t \mid$$

$$= O(\sqrt{q}\, t \exp(-aq^{2\gamma-1})) + O(q^{(3-2\gamma/2)} t \exp(-aq^{2\gamma-1})).$$

Hence the lemma follows in view of boundedness of $\phi_x\ (t)$ and since $\sin\left(\frac{t}{2}\right) > \left(\frac{t}{\pi}\right)$ $(0 < t < \pi)$.

Lemma 2 – Let $f \in C[0, 2\pi]$ such that $\omega_f \leqslant \omega$, where ω is as in section 1, and let condition (1.12) be satisfied. Then

$$\int_0^\pi \frac{\phi_x(t)}{\sin t/2} \exp(-mt^2/4a) \sin(m+\tfrac{1}{2}) t\, dt = O\, \omega\, (\frac{1}{m})).$$

PROOG : If u and v are as defined in (1.14) we write

$$\int_0^\pi \frac{\phi_x(t)}{\sin t/2} \exp(-mt^2/4a) \sin(m+\tfrac{1}{2}) t\, dt$$

$$= \left(\int_0^{u(m)} + \int_{u(m)}^{v(m)} + \int_{v(m)}^{\pi}\right) \frac{\phi_x(t)}{\sin\ t/2} \exp(-mt^2/4a) \sin(m+\tfrac{1}{2}) t\, dt$$

$$= J_1 + J_2 + J_3, \text{ say.} \qquad \ldots(2.3)$$

Now

$$|J_1| \leqslant \pi \int_0^{u(m)} \frac{|\phi_x(t)|}{t} \exp(-mt^2/4a) \sin(m+\tfrac{1}{2}) t \mid dt$$

$$\leqslant \pi \int_0^{u(m)} \frac{\omega(t)}{(t)} (m+\tfrac{1}{2}) t\, dt$$

$$\leqslant \pi^2\, \omega\, (u(m)) = O(\omega(\frac{1}{m})). \qquad \ldots(2.4)$$

Using (2.2) we estimate J_3:

$$| J_3 | \leqslant \pi \int_{\nu(m)}^{\pi} \frac{| \phi_x (t) |}{t} \exp(-mt^2/4a)\, dt$$

$$= O\{ \int_{\nu(m)}^{\infty} 1/t \exp(-mt^2/4a)\, dt\}$$

$$= O\left\{ \int_{\frac{m[\nu(m)]^2}{4a}}^{\infty} \frac{1}{z} e^{-z}\, dz \right\}$$

$$= O\{m^{2\eta-1} \exp(-\frac{\pi^{2\eta}}{4a} m^{1-2\eta})\}. \qquad \ldots(2.5)$$

Now, for $\alpha > 0$, $A > 0$, λ and δ any real constants we have

$$m^{\lambda} \exp(-Am^{\alpha}) = O(m^{\delta}). \qquad \ldots(2.6)$$

Also, by the condition (1.10) on ω,

$$q^{-1/2}, m^{-1/2} = O(\omega(\frac{1}{m})). \qquad \ldots(2.7)$$

Thus it is enough to show that

$$| J_2 | = O(\omega(\frac{1}{m})).$$

Since

$$\operatorname{cosec} t - \frac{1}{t} = O(t)$$

with (1.10), we have

$$J_2 = \int_{u(m)}^{v(m)} \frac{\phi_x(t)}{\sin \frac{1}{2} t} \exp(-mt^2/4a) \sin(m + \tfrac{1}{2} t)\, dt$$

$$= 2 \int_{u(m)}^{v(m)} \frac{\phi_x(t)}{t} \exp(-mt^2/4a) \sin(m + \tfrac{1}{2})\, t\, dt$$

$$+ \int_{u(m)}^{v(m)} \phi_x(t) (\operatorname{cosec} \frac{t}{2} - \frac{2}{t}) \exp(-mt^2/4a) \sin(m+\tfrac{1}{2})t\, dt$$

(*equation continued on p. 375*)

$$= 2 \int_{u(m)}^{v(m)} \frac{\phi_x(t)}{t} \exp(-mt^2/4a) \sin(m + \tfrac{1}{2})\, t\, dt + O(\omega \frac{1}{m}))$$

$$= J_4 + O(\omega(\frac{1}{m})), \text{ say.} \qquad \ldots(2.8)$$

We shall write J_4 as follows :

$$J_4 = \int_{u(m)}^{v(m)} \frac{\phi_x(t)}{t} \exp(-mt^2/4a) \sin(m + \tfrac{1}{2})\, t\, dt$$

$$- \int_{0}^{v(m)-u(m)} \frac{\phi_x(t + u(m))}{t + u(m)} \exp(-m(t + u(m))^2/4a)$$

$$\times \sin(m + \tfrac{1}{2})\, t\, dt$$

$$= \int_{u(m)}^{v(m)} \frac{\phi_x(t) - \phi_x(t + u(m))}{t} \exp(-mt^2/4a) \sin(m + \tfrac{1}{2})\, t\, dt$$

$$+ \int_{u(m)}^{v(m)} \frac{\phi_x(t + u(m))}{t} [\exp(-mt^2/4a)$$

$$- \exp(-m(t + u(m))^2/4a)] \sin(m + \tfrac{1}{2})\, t\, dt$$

$$+ \int_{u(m)}^{v(m)} \phi_x(t + u(m)) \exp(-m(t + u(m))^2/4a) \left[\frac{1}{t}\right.$$

$$\left. - \frac{1}{t + u(m)}\right] \sin(m + \tfrac{1}{2})\, t\, dt$$

$$- \int_{0}^{u(m)} \frac{\phi_x(t + u(m))}{t + u(m)} \exp(-m(t + u(m))^2/4a) \sin(m + \tfrac{1}{2})\, t\, dt$$

$$+ \int_{v(m)-uu(m)}^{v(m)} \frac{\phi_x(t + u(m))}{t + u(m)} \exp(-m(t + u(m))^2/4a) \sin$$

$$(m + \tfrac{1}{2})\, t\, dt = I_1 + I_2 + I_3 + I_4 + I_5.$$

By hypothesis

$$I_1 = O\ (\omega\ (\frac{1}{m}\ )). \qquad \ldots(2.9)$$

By mean value theorem

$$\exp\ (-mt^2/4a) - \exp\ (-m(t+u(m))^2/4a) = \frac{2u\ (m)m\theta}{4a} \exp\ (-m\theta^2/4a)$$

for some θ such that $t < \theta < t + u\ (m) < 2t$. Hence

$$\exp\ (-mt^2/4a) - \exp\ (-m\ (t + u\ (m))^2/4a) = O\ (t \exp\ (-mt^2/4a)).$$

Thus

$$|\ I_2\ | = O\ \{\int_{u(m)}^{v(m)} |\ \phi_x\ (t + u\ (m))\ |\ \exp -mt^2/4a)\ dt\}.$$

By (1.10) and (1.11), we have

$$|\ I_2\ | \ = O\ \{\sqrt{m}\ \omega\ (\frac{1}{m}\ ) \int_{n(m)}^{v(m)} t^{1/2} \exp\ (-mt^2\ /4a)\ dt\}$$

$$= O\ (\omega\ (\frac{1}{m})). \qquad \ldots(2.10)$$

Again, by (1.10) and (1.11) we have

$$|\ I_3\ | \leqslant u\ (m) \int_{u(m)}^{v(m)} \frac{|\ \phi_x\ (t + u\ (m))\ |}{t\ (t + u\ (m))}\ dt$$

$$= O\ \{u\ (m) \sqrt{m}\ \omega\ (\frac{1}{m}\ ) \int_{u(m)}^{v(m)} \frac{1}{t\ (t + u\ (m))^{1/2}}\ dt\}$$

$$= O\ (\omega\ (\frac{1}{m}\ )). \qquad \ldots(2.11)$$

Also,

$$|\ I_4\ | \ \leqslant \int_0^{u(m)} \frac{|\ \phi_x\ (t + u\ (m))\ |}{t + u\ (m)}\ |\ \sin\ (m + \tfrac{1}{2})\ t\ |\ dt$$

$$\leqslant \int_{u(m)}^{2u(m)} \frac{|\ \phi_x\ (t)\ |}{t}\ (m + \tfrac{1}{2})\ t\ dt$$

(equation continued on p. 377)

$$\leqslant (m + \tfrac{1}{2}) \int_{u(m)}^{2u(m)} \omega(t)\, dt$$

$$\leqslant \pi\, \omega(2u(m)) = O(\omega(\frac{1}{m})). \qquad \ldots(2.12)$$

Finally,

$$| I_5 | \leqslant \int_{v(m)-u(m)}^{v(m)} \frac{| \phi_x(t + u(m)) |}{t + u(m)}\, dt$$

$$= \int_{v(m)}^{v(m)+u(m)} \frac{| \phi_x(t) |}{t}\, dt.$$

By (1.10) and (1.11) we have

$$| I_5 | = O\{\sqrt{M}\, \omega(\frac{1}{m} \int_{v(m)}^{v(m)+u(m)} t^{-1/2}\, dt\}.$$

But

$$[v(m) + u(m)]^{1/2} - [v(m)]^{1/2} = [v(m)]^{1/2} \left\{(1 + \frac{u(m)}{v(m)}^{1/2} - 1\right\}.$$
$$= O\{[u(m)]^{1/2}\}.$$

Thus

$$| I_5 | = O(\omega(\frac{1}{m})). \qquad \ldots(2.13)$$

Hence the lemma follows from (2.3) to (2.13).

Lemma 3—Let $0 < \beta < 1$, $0 < t \leqslant \pi$ and let ρ, θ, t and β satisfy

$$\rho_{\theta}^{-i\theta} = 1 - \beta e^{-it} \qquad \ldots(2.14)$$

Then, for $p = 0, 1, 2, \ldots$,

$$\text{(i)} \quad | \theta - \frac{\beta t}{1 - \beta} | \leqslant C t^3 \qquad \ldots(2.15)$$

$$\text{(ii)} \quad (\frac{1 - \beta}{\rho})^p \leqslant \text{exy}\,(-Apt^2) \qquad \ldots(2.16)$$

where C and A are positive constants depending on β, and

$$\text{(iii)} \quad \left(\frac{1 - \beta}{\rho} \right)^p - \exp\left(\tfrac{1}{2} p\beta\left(\frac{t}{1-\beta}\right)^2\right) = O(pt^4). \qquad \ldots(2.17)$$

These results are known. For example, for (i) see Miracle[7] and for (ii) and (iii) see Forbes[2].

3. Proof of the Results

Proof of Theorem 1—By (1.3), (1.4) and (1.8) we get

$$\sigma_p(x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \frac{\phi_x(t)}{\sin \frac{1}{2}t} \sum_{k=0}^{\infty} c_k(p) \sin (k+\tfrac{1}{2})\, t\, dt + O(q^{-1/2})$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\phi_x(t)}{\sin \frac{1}{2}t}\Big[\Big(\sum_{|k-q| \leqslant q^{\gamma}} + \sum_{|k-q| > q^{\gamma}}\Big)$$

$$\times c_k(p) \sin (k+\tfrac{1}{2})\, t\Big]\, dt + O(q^{-1/2})$$

$$= S_1 + S_2 + O(q^{-1/2}). \qquad \ldots(3.1)$$

We first estimate S_2 :

$$| S_2 | \leqslant \int_0^{\pi} \frac{|\phi_x(t)|}{t} \sum_{|k-q| > q^{\gamma}} c_k(p)\, (k+\tfrac{1}{2})\, t\, dt \qquad \ldots(3.2)$$

$$= O(\exp(-q^{\mu}))\ (\text{by } (1.7)).$$

Using (1.6), S_1 can be written as follows :

$$S_1 = \frac{1}{\pi}\int_0^{\pi} \frac{\phi_x(t)}{\sin \frac{1}{2}t} \sum_{|k-q| < q^{\gamma}} g(q,k) \Big\{1 + O\Big(\frac{|k-q|+1}{q}\Big)$$

$$+ O\Big(\frac{|k-q|^3}{q^2}\Big)\Big\} \sin (k+\tfrac{1}{2})\, t\, dt$$

$$= S_3 + S_4 + S_5, \text{ say.} \qquad \ldots(3.3)$$

We estimate S_4 and S_5 by splitting the integral into two parts and using condition (1.10) and properties of sine function.

$$| S_3 | \leqslant \Big(\int_0^{1/q} + \int_{1/q}^{\pi}\Big) \frac{\phi_x(t)}{t} \sum_{|k-q| \leqslant q^{\gamma}} g(d,k)\, O\Big(\frac{|k-q|+1}{q}\Big)$$

$$\times \sin (k+\tfrac{1}{2})\, t\, dt$$

(*equation continued on p.* 379)

$$= O\left\{\int_0^{1/q} |\phi_x(t)| \sum_{|k-q| \leqslant q^\gamma} g(q, k)\, O\left(\frac{|k-q|+1}{q}\right)\right.$$

$$\left.\times\, (|k-q| + q + \tfrac{1}{2})\, dt\right\}$$

$$+ O\left\{\int_{1/q}^{\pi} \frac{|\phi_x(t)|}{t} \sum_{|k-q| \leqslant q^\gamma} g(q, k)\left(\frac{|k-q|+1}{q}\right)\right.$$

$$\left. dt\right\}$$

$$= O\left\{\sqrt{q} \int_0^{1/q} |\phi_x(t)|\, dt\right\} + O\left\{\frac{1}{\sqrt{q}} \int_{1/q}^{\pi} \frac{|\phi_x(t)|}{t}\, dt\right\}$$

$$= O\left(\omega\left(\frac{1}{q}\right)\right). \qquad \ldots(3.4)$$

Similarly,

$$|S_5| = O(\omega(\frac{1}{q})). \qquad \ldots(3.5)$$

Since $m = m(p)$ is an integer valued function of p, by Lemmas 1 and 2 we have

$$\frac{1}{\pi} \int_0^{\pi} \frac{\phi_x(t)}{\sin \frac{1}{2}t} \sum_{|k-m| \leqslant m^\gamma} g(m, k) \sin(m + \tfrac{1}{2})\, t\, dt$$

$$= \frac{1}{\pi} \int_0^{\pi} \frac{\phi_x(t)}{\sin \frac{1}{2}t} \exp(-mt^2/4a) \sin(m + \tfrac{1}{2})\, t\, dt$$

$$+ O\, m^{3/2-\gamma} \exp(-am^{2\gamma-1}))$$

$$= O(\omega\, \frac{1}{m})). \qquad \ldots(3.6)$$

Now we shall estimate the difference :

$$\int_0^{\pi} \frac{\phi_x(t)}{\sin \frac{1}{2}t} \sum_{|k-q| \leqslant q^\gamma} g(q, k) \sin(k + \tfrac{1}{2})\, t\, dt$$

(equation continued on p. 380)

$$- \int_0^\pi \frac{\phi_x(t)}{\sin \frac{1}{2}t} \sum_{|k-m| \leqslant m^\gamma} g(m, k) \sin (k + \tfrac{1}{2}) t \, dt$$

$$= \int_0^\pi \frac{\phi_x(t)}{\sin \frac{1}{2}t} \sum_{m \leqslant k < m + m^\gamma} (g(q, k) - g(m, k)) \sin (k + \tfrac{1}{2}) t \, dt$$

$$+ \int_0^\pi \frac{\phi_x(t)}{\sin \frac{1}{2}t} \sum_{m - m^\gamma \leqslant k < m} (g(q,k) - g(m,k)) \sin (k + \tfrac{1}{2}) t \, dt$$

$$- \int_0^\pi \frac{\phi_x(t)}{\sin \frac{1}{2}t} \sum_{q + q^\gamma < k < m + m^\gamma} g(q, k) \sin (k + \tfrac{1}{2}) t \, dt$$

$$+ \int_0^\pi \frac{\phi_x(t)}{\text{n} \frac{1}{2}t} \sum_{q - q^\gamma \leqslant k \leqslant m - m^\gamma} g(q, k) \sin (k + \tfrac{1}{2}) t \, dt$$

$$= D_1 + D_2 + D_3 + D_4. \qquad \ldots(3.7)$$

First,

$$| D_3 | \leqslant \int_0^\pi \frac{| \phi_x(t)}{t} \sum_{q + q^\gamma < k \leqslant m + m^\gamma} g(q, k) \, | \sin (k + \tfrac{1}{2}) t \, | \, dt$$

$$= O \{ \sqrt{q} \exp (aq^{2\gamma - 1}) \}. \qquad \ldots(3.8)$$

Similarly

$$| D_4 | = O \{ \sqrt{q} \exp (-aq^{2\gamma - 1}) \}. \qquad \ldots(3.9)$$

For $q < m \leqslant k \leqslant m + m^\gamma$ we have

$$O \leqslant (k - m)/\sqrt{m} < (k - q)/\sqrt{q} < (k - n)/\sqrt{n}.$$

Now (cf. Ikeno[4], p.259)

$$| g(q, k) - g(m, k) | = O \left\{ g(m, k) \left(\frac{(k - m)^2}{m^2} + \frac{| k - m |}{m} + \frac{1}{m} \right) \right\}. \qquad \ldots(3.10)$$

Using (3.10) and the condition (1.10) on ω, we get

$$| D_1 | = O \left\{ \int_0^\pi \frac{| \phi_x(t) |}{t} \sum_{m \leqslant k \leqslant m + m^\gamma} g(m, k) \left(\frac{(k - m)^2}{m^2} \right. \right.$$

(equation contd. on p. 381)

$$+ \frac{|k-m|}{m} + \frac{1}{m}\Big) \times |\sin(k+\tfrac{1}{2})t|\,dt\Big\}$$

$$= O\Big\{\int_0^{1/m} \frac{|\phi_x(t)|}{t} \sum_{m \leqslant k \leqslant +m^\gamma} g(m,k)\Big(\frac{(k-m)^2}{m^2}$$

$$+ \frac{|k-m|}{m} + \frac{1}{m}\Big)(|k+m|+m+\tfrac{1}{2})\,t\,dt\Big\}$$

$$+ O\Big\{\int_{1/m}^{\pi} \frac{|\phi_x(t)|}{t} \sum_{m \leqslant k \leqslant m+m^\gamma} g(m,k)\Big(\frac{(k-m)^2}{m^2}$$

$$+ \frac{|k-m|}{m} + \frac{1}{m}\Big)dt\Big\}$$

$$= O\Big\{\sqrt{m}\int_{1/m}^{1/m} |\phi_x(t)|\,dt\Big\} + O\Big\{\frac{1}{\sqrt{m}}\int_{1/m}^{\pi} \frac{|\phi_x(t)|}{t}\,dt\Big\}$$

$$= O(\omega(\frac{1}{m})). \qquad \ldots(3.11)$$

If $k \leqslant n < q < m$, we have

$$(k-m)/\sqrt{m} < (k-q)/\sqrt{q} < (k-n)\sqrt{n} \leqslant 0$$

and (cf. Ikeno[4], p. 260)

$$|g(q,k) - g(m,k)| = O\Big\{g(n,k)\Big(\frac{(k-n)^2}{n^2} + \frac{|k-n|}{n} + \frac{1}{n}\Big)\Big\}.$$

Using this and arguing analogous to the estimation of D_1 we get

$$|D_2| = O(\omega(\frac{1}{m})). \qquad \ldots(3.12)$$

Theorem 1 follows from (3.1) to (3.12) in view of (2.6) and (2.7).

Proof of Theorem 2—With ρ and θ defined in (2.14) we have

$$S^p_\beta(x) - f(x) = \frac{1}{\pi}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{1}{2}t} \sum_{k=0}^{\infty} (1+\beta)^{p+1}\binom{p-k}{p}\beta^k$$

$$\times \sin(k+\tfrac{1}{2})\,t\,dt$$

$$= \frac{1}{\pi}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{1}{2}t}\,\mathrm{Im}\Big\{e^{it/2}\frac{(1-\beta)^{p+1}}{(1-\beta e^{it})^{p+1}}\Big\}dt$$

(*equation continued on p.* 282)

$$= \frac{1}{\pi} \int_0^{\pi} \frac{\phi_x(t)}{\sin \frac{1}{2}t} \left(\frac{1-\beta}{\rho} \right)^{p+1} \sin ((p+1)\,\theta + \tfrac{1}{2}t)\, dt$$

$$= \frac{1}{\pi} \left(\int_0^{a(p)} + \int_{a(p)}^{b(p)} + \int_{b(p)}^{\pi} \right) \frac{\phi_x(t)}{\sin \frac{1}{2}t} \left(\frac{1-\beta}{\rho} \right)^{p+1}$$

$$\times \sin ((p+1)\,\theta + \tfrac{1}{2}t)\, dt$$

$$= \chi_1 + \chi_2 + \chi_3. \qquad \ldots(3.13)$$

Now, using the fact that $(1 - \beta) \leqslant \rho$ as also lemma 3 (i), we have

$$| \chi_1 | = O \left\{ \int_0^{a(p)} \frac{| \phi_x(t) |}{t} \left(\tfrac{1}{2}t + (p+1)\left(Ct^3 + \frac{\beta t}{1-\beta}\right)\right) dt \right\}$$

$$= O \{ p \int_0^{a(p)} t^{\alpha}\, dt \}$$

$$= O(p^{-\alpha}).$$

By applying Lemma 3 (ii) to χ_3 and arguing similar to the estimation of J_3 in Lemma 2, we get

$$| \chi_3 | = O \left\{ \int_{b(p)}^{\pi} \frac{| \phi_x(t) |}{t} \exp (-A\,(p+1)\,t^2)\, dt \right\}$$

$$= O \left\{ \int_{b(p)}^{\infty} \frac{1}{t} \exp (-A\,(p+1)t^2)\, dt \right\}$$

$$= O \left\{ \int_{A(p+1)[b(p)]^2}^{\infty} \frac{1}{u}\, e^{-u}\, du \right\}$$

$$= O \{ p^{2\delta-1} \exp (-A\, p + 1)\, [b\,(p)\}^2) \}$$

$$= O(p^{-\alpha}).$$

χ_2 can be estimated exactly like the 'η_2' in the context of Taylor means treated by Chui and Holland[1] (pp. 35-37). We omit the details.

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A MODEL FOR MICROPOLAR FLUID FILM MECHANISM WITH REFERENCE TO HUMAN JOINTS

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This paper analyses the variation of pressure and load capacity with reference to load bearing human joints by introducing a continuously varying porosity model for lower plate instead of usual uniform mono or multi-layered models studied so far. A micropolar fluid film is taken as a lubricant. By suitable choice of non-dimensional porosity variation parameter $\bar{\alpha}$; it is shown that trends of variations are fairly in agreement with those recorded in earlier investigations.

NOTATION

a = Characteristic length of bearing,

β = film thickness,

β_0 = initial film thickness,

$\bar{\beta}$ = dimensional film thickness, β/β_0

α = porosity parameter of variation

$\bar{\alpha}$ = non-dimensional porosity parameter of variation, $a\alpha$

H = thickness of cartilage,

K_0 = porosity of cartilage,

$\bar{K}_0$ = dimensionless porosity parameter, K_0/β_0^2

l = $(\gamma/4\mu)^{1/2}$

L = β_0/l

N = coupling number $(\chi/(2\mu + \chi))^{1/2}$

p = pressure in fluid film region

$\bar{p}$ = non-dimentional pressure in fluid film region, $2p\,\beta_0^3\,/\mu v a^2$,

u, v = velocity components in fluid film region,

$\bar{u}, \bar{v}$ = velocity components in porous matrix,

v = velocity of approach,

W = load capacity,

$\bar{W}$ = dimensionless load capacity $3W\beta_0^2/\mu va^3$,

x = x-coordinate

y = y-coordinate

μ = Newtonian viscosity coefficient

γ, χ = viscosity coefficient for micropolar fluid,

ν = micro-rotation velocity.

1. Introduction

Within the last five decades sufficient thought has been given to the study of lubrication mechanism in human joints but the recent studies have brought out a fairly clear picture of this process. The human joint may be visualised as a class of mechanical bearing because the fluid in the cavity between two mating bones is believed to act as lubricant. The human joints in this context may be described as a system consisting of two mating bones covered with cartilage with synovial fluid between them. Jones[1] observed that a fluid film region was predominant mode of lubrication mechanism. The studies of Ogston and Stanier[2] have brought out the visco-elastic character of the synovial fluid. Various types of lubrication mechanism are believed to occur in the human joints like hydrodynamic[3], boundary[4], weeping[5] and mixed lubrication[6]. Dintenfass[7] found that synovial fluid is non-Newtonian due to the presence of hyaluronic acid (a long chain polymer) molecules and its viscosity decreases with increasing shear rate. This view was experimentally supported by Bloch and Dintenfass[8], Maroudas[9] and Dowson[10]. Further more the studies of Dowson[10] and Mow[11] confirmed that the synovial fluid acts as lubricant. Eringen[12] formulated the theory of micropolar fluids, which has been used by many authors under various physical situations. As the micropolar fluid may be considered for polymers, it can be taken for the synovial fluid consisting a long chain polymers.

Cartilage is basically a two-phase deformable porous material which can absorb or give out fluid owing to the established pressure gradient by either squeeze film action of the synovial fluid or consolidation of the solid matrix by tissue deformation. The studies of Clark[13], Torizilli and Mow[14,15] have pointed out that cartilage is a three layered porous medium consisting of a superficial tangential zone, a middle zone and a deep zone. Nigam *et al.*[16] investigated the effect of the variation of porosity in the

upper most layer of the cartilage which according to them plays a predominant role in the self adjusting nature of the human joint, taking a three layered porous medium. Sinha[17] has considered the problem of lubrication of two approaching surfaces, one of which is covered with a layer of porous material and investigated the influence of porosity of the cartilage, film thickness, the thickness of the porous bad on axial pressure and load bearing capacity. He[18] also investigated the influence of magnetic field on squeeze film lubrication with reference to human joints and noted that magnetotherapy can be of significant use in the treatment of diseases of human joints. Recently Tandon and Rakesh[19] studied the lubrication mechanism occurring in knee joint replacement under restricted motion.

In this paper, the superficial division of cartilage matrix into three distinct layers is replaced by a continuously varying porosity matrix. The proposed model assumes flow of a fluid in a porous matrix of continuously varying porosity with a squeeze film of a micropolar fluid between two approaching surfaces. The mathematical analysis of the problem has been done by taking the continuity of pressure and velocities at the inter-face of the fluid film and the porous layer.

2. Mathematical Formulation

Referring to Fig. 1, the proposed model is conceived as a flow model of a squeeze film lubrication between two approaching surfaces with micropolar fluid and flow of

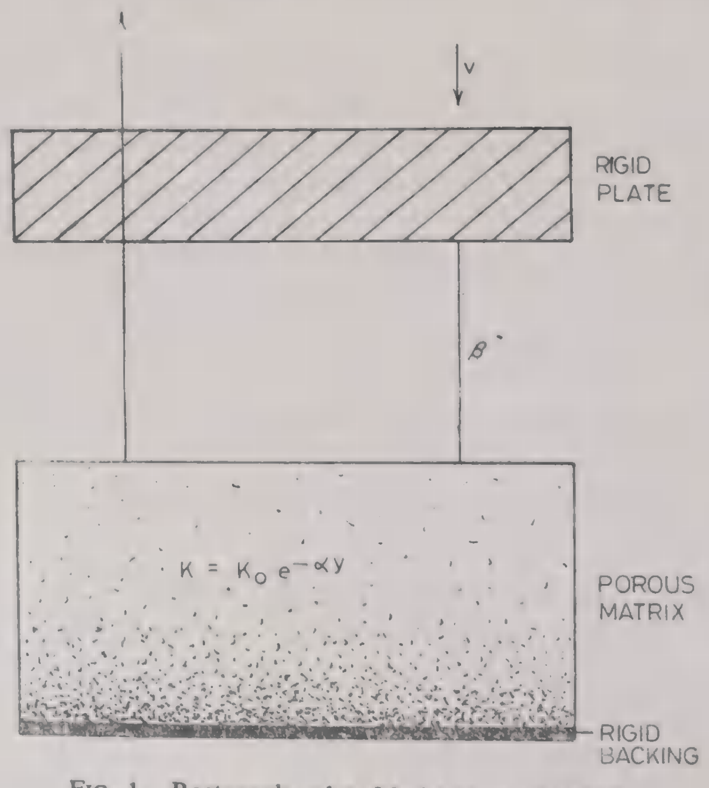

Fig. 1. Rectangular plate Model (synovial joint).

viscous fluid in a continuous porous matrix with variable porosity. The porous matrix is fixed and upper surface is rigid and moves with uniform velocity V in the negative direction of y-axis as shown in the Fig. 1.

2.1 *Fluid Film Region*

Following Eringen[12] the field equations for micropolar fluid may be reduced to the following form,

$$\frac{\partial^2 \nu}{\partial y^3} - m^3 \frac{\partial \nu}{\partial y} = \frac{m^2}{2\mu} \frac{\partial p}{\partial x} \qquad \ldots(1)$$

$$0 = \gamma \frac{\partial^2 \nu}{\partial y^2} - 2x\nu - \chi \frac{\partial u}{\partial y} \qquad \ldots(2)$$

and equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial \nu}{\partial y} = 0 \qquad \ldots(3)$$

where

$$m = \frac{N}{l} = \left(\frac{4\mu\chi}{\nu(2\mu+\chi)}\right)^{1/2}$$

$$N = \left(\frac{\chi}{2\mu+\chi}\right)^{1/2} \text{ and } l = \left(\frac{\gamma}{4\mu}\right)^{1/2}.$$

2.2. *Porous Region*

Following Darcy's law the flow of a viscous fluid in a porous matrix is governed by

$$\bar{u} = -\frac{K}{u}\frac{d\bar{p}}{dx} \qquad \ldots(4)$$

$$\bar{v} = -\frac{K}{\mu}\frac{d\bar{p}}{dy} \qquad \ldots(5)$$

and equation of continuity is

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \qquad \ldots(6)$$

where

$$K = K_0 e^{-\alpha y}.$$

2.3. *Boundary conditions for Fluid Film*

The boundary conditions is the fluid film region are

$$\left.\begin{aligned} p &= 0 \text{ at } x = \pm a \\ v &= 0 \text{ at } y = 0, y = \beta \\ u &= 0 \text{ at } y = \beta \\ u &= \bar{u} \text{ at } y = 0 \\ v &= \bar{v} \text{ at } y = 0 \\ \text{and} \quad v &= V \text{ at } y = \beta \end{aligned}\right\} . \qquad \ldots(7)$$

2.4. *Boundary Conditions for porous matrix*

The boundary condition for porous matrix are

$$\bar{p} = 0 \text{ at } x = \pm a \qquad \ldots(8)$$

and

$$\frac{\partial \bar{p}}{\partial y} = 0 \text{ at } y = -H. \qquad \ldots(9)$$

2.5 *Matching Condition*

On the interface, $y = 0$,

$$p(x) = \bar{p}(x, 0). \qquad \ldots(10)$$

3. Solutions

The geometry of the model and the frames of reference are shown in Fig. 1. In what follows the conventional assumptions of lubricant theory are assumed.

3.1. *Porous region*

The lubricant in the porous region is an incompresible Newtonian fluid. Using the value of $\bar{u}$ and $\bar{v}$ from eqns. (4) and (5) in eqn. (6) we get

$$\frac{\partial^2 \bar{p}}{\partial x^2} + \frac{\partial^2 \bar{p}}{\partial y^2} - \alpha \frac{\partial \bar{p}}{\partial x} = 0. \qquad \ldots(11)$$

Integrating eqn. (10) with respect to y over the porous matrix thickness H, we get

$$\left(\frac{\partial \bar{p}}{\partial y}\right)_{y=0} = -H \frac{\partial^2 \bar{p}}{\partial x^2} + \alpha H \frac{\partial \bar{p}}{\partial x} \qquad \ldots(12)$$

assuming that H is small.

3.2. *Fluid Film Region* of eqn. (1) under boundary conditions $v = 0$ at $y = 0, \beta$, is

$$v = -\frac{C}{\sinh m\beta}\{\sinh m\beta\,(\beta - y) - \sinh m\beta + \sinh my\}$$

$$+ \frac{1}{2\mu}\frac{dp}{dx}\left\{\frac{\beta \sinh my}{\sinh m\beta} - y\right\}. \qquad \ldots(13)$$

Substituting this value of ν in equation (2), we get

$$\frac{\partial u}{\partial y} = \frac{y}{\mu}\frac{dp}{dx} - \frac{Dm\beta \sinh my}{2\mu \sinh m\beta}\frac{dp}{dx}$$

$$- \frac{C}{\sinh m\beta}\{2 \sinh m\beta - Dm \sinh m(\beta - y) - Dm \sinh my\} \quad ...(14)$$

where

$$D = \frac{2}{m} - \frac{\gamma m}{\chi}.$$

Integrating eqn. (14) with respect to y, under boundary conditions we get

$$u = \frac{y^2}{2\mu}\frac{dp}{dx} - \frac{K_0}{\mu}\frac{dp}{dx} + \frac{D\beta}{2\mu}\frac{dp}{dx}\frac{(1 - \cosh my)}{\sinh m\beta}$$

$$- \frac{C}{\sinh m\beta}\{2y \sinh m\beta + D \cosh m(\beta - y) - D \cosh my$$

$$- D(\cosh m\beta - 1) \quad ...(15)$$

where

$$C = -\frac{\frac{\beta}{2\mu}\frac{dp}{dx}\{\sinh m\beta\left(\beta - \frac{2K_0}{\beta}\right) - D(\cosh m\beta - 1)}{2\{D(\cosh m\beta - 1) - \beta \sinh m\beta\}}.$$

Integrating eqn (3) with respect to y we have

$$\frac{\partial}{\partial x}\int_0^\beta u dy = -\frac{K_0}{\mu}\left\{- H\frac{d^2p}{dx^2} + \alpha H\frac{dp}{dx}\right\}_\xi + V. \quad ...(16)$$

Substituting the value of from eqn. (15) in eqn. (16) we get

$$\frac{d^2p}{dx^2} + a_0\frac{dp}{dx} = b_0 \quad ...(17)$$

where

$$a_0 = -\frac{12\alpha H K_0}{\beta\left\{\beta^2 + 6K_0 + \frac{12HK_0}{\beta} + \frac{6D}{m} - 3D\beta \coth\frac{m\beta}{2}\right\}}$$

and

$$b_0 = -\frac{12\mu v}{\beta\left\{\beta^2 + 6K_0 + \frac{12HK_0}{\beta} + \frac{6D}{m} - 3D\beta \coth\frac{m\beta}{2}\right\}}.$$

On solving eqn (17) under boundary conditions, we get

$$p = \frac{b_0 a}{a_0}\left\{- \coth a_0 a + \frac{\exp\left(-a_0 a \frac{x}{a}\right)}{\sinh a_0 a} + \frac{x}{a}\right\}. \qquad \ldots(18)$$

3.3 *Load*

The load carrying capacity of the joint is given by

$$W = \int_{-a}^{a} p dx$$

$$= - \frac{2b_0 a}{a_0}\left\{ a \coth a_0 a = \frac{1}{a_0 a} \right\}. \qquad \ldots(19)$$

4. Non-dimensionalization

Using following non-dimensional parameter

$$\bar{x} = \frac{x}{a}, \bar{\beta} = \frac{\beta}{\beta_0}, L = \frac{\beta_0}{l}, \bar{K}_0 = \frac{K_0}{\beta_0{}^2}, \bar{H} = \frac{H}{\beta_0},$$

$$\bar{p} = \frac{2p\beta_0^3}{\mu v a^2}, \bar{W} = \frac{3W\beta_0^3}{\mu v a3} \bar{\alpha} = a\,\alpha, \bar{a} = a_0 a, \bar{b} = \frac{b_0 \beta_0^3}{\mu v}.$$

The non-dimensional pressure distribution is given by

$$\bar{p} = \frac{2\bar{b}}{\bar{a}}\left\{ - \coth \bar{a} + \frac{e^{\bar{a}\bar{x}}}{\sinh \bar{a}} + \bar{x} \right\}. \qquad \ldots(20)$$

The non-dimensional load carrying capacity is

$$\bar{W} = \frac{4\bar{b}}{\bar{a}}\left\{ \frac{1}{a} - \coth \bar{a} \right\} \qquad \ldots(21)$$

where

$$\bar{a} = - \frac{12\,\bar{\alpha}\,\bar{H}\bar{K}_0}{\bar{\beta}\left\{\bar{\beta}^2 + 6\bar{K}_0 + \frac{12\,\bar{K}_0\bar{H}}{\bar{\beta}} + \frac{12}{L^2} - \frac{6N\bar{\beta}}{L} \coth \frac{NL\bar{\beta}}{2}\right\}}$$

and

$$\bar{b} = - \frac{12}{\bar{\beta}\left\{\bar{\beta}^2 + 6\bar{K}_0 + \frac{12\,\bar{K}_0\bar{H}}{\bar{\beta}} + \frac{12}{L^2} - \frac{6N\bar{\beta}}{L} \coth \frac{NL\bar{\beta}}{2}\right\}}$$

5. Discussion

To bring out the effects of change of shape and size of micromolecules and the concentration on bearing characteristics by using micropolar fluid as lubricant (as a substitute of synovial fluid) in a conjunction with influence of the porous nature of the cartilage. The micropolar fluid involves two dimensionless parameters N and L. Which do not occur in Newtonian theory and are called coupling number and material characteristic length respectively. In limiting cases as $N \to 0$ or $N \to \infty$, they consides with Newtonian cases provided μ is replaced bp $(\mu + \frac{1}{2}\chi)$. Also a large values of l and small values of β_0 give rise to increase the effective viscosity and this is in line with the experimental evidence of Hanniker[20]. Consequently the number L gives the influence of the shape and size of the suspended particles and N is a measure of the concentration of suspended particles.

Torzilli and Mow[14,15] have given the following data of articular cartilage by considering a three layered model for the cartilage

$$\frac{K_1}{\mu} = 3 \times 10^{-13} \text{ cm}^4/\text{dyne} - \text{sec}$$

$$\frac{K_2}{\mu} = 6 \times 10^{-13} \text{ cm}^4/\text{dyne} - \text{sec}$$

$$\frac{K_3}{\mu} = 9 \times 10^{-13} \text{ cm}^4/\text{dyne} - \text{sec}$$

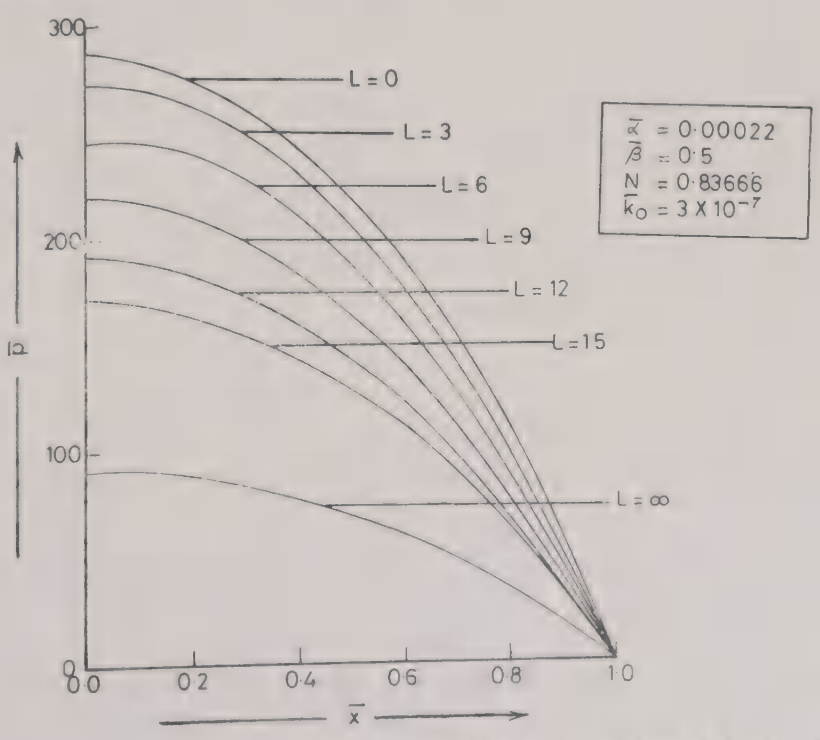

Fig. 2. Non dimensional axial pressure distribution for different values of shape and size index L.

$H_1 = 200$ microne, $H_2 = 0.2$ cm, $H_3 = 0.28$ cm and $\mu = 1$ poise.

Following Nigam *et al.*[16] minimum film thickness $\beta_0 = 10^{-3}$ and upper surface porosity of the cartilage $K_0 = 3 \times 10^{-7}$. We have by assuming continuous variation in the porosity of the cartilage taken $H_1 + H_2 + H_3$ is 0.5 cm and have introduced a non-dimensional porosity parameter of variation $\bar{\alpha} = 0.00022$. Since exact data for quantities N and L for synovial fluid is not known we have chosen some values for these quantities.

In Fig. 2 the variation of non-dimensional axial pressure $\bar{p}$ is shown for different values of shape and size index L it is obvious that as L increases, axial pressure decreases i. e. when shape and size of hyaluronic acid molecules decreases, axial pressure also decreases.

Figure 3 shows that as porosity at the interface increases, pressure decreases and this happens at a faster rate. The case of $\bar{K}_0 = 0$ corresponds to the case of non-

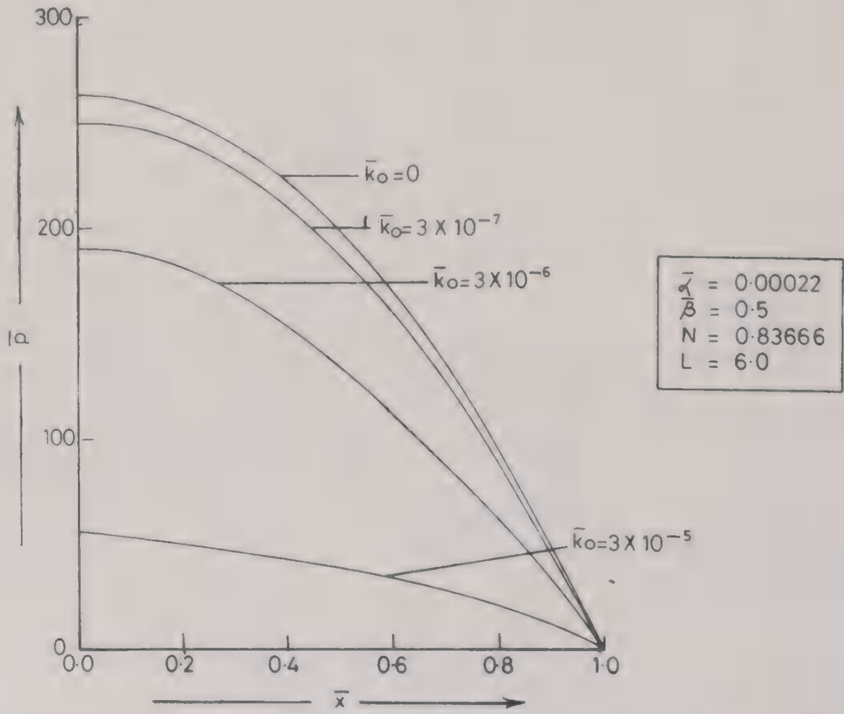

FIG. 3. Non dimensional axial pressure distribution for different values of porosity of superficial tangential layer.

porous cartilage for which the pressure is maximum. These results are in agreement with the observation of Prakash and Sinha[21].

In Fig. 4 the variation of non-dimensional load bearing capacity $\bar{W}$ with the film thickness $\bar{\beta}$ for diffeaent values of the porosity of the interface is shown. It is noted that the increase of film thickness, decreases the load bearing capacity $\bar{W}$ and

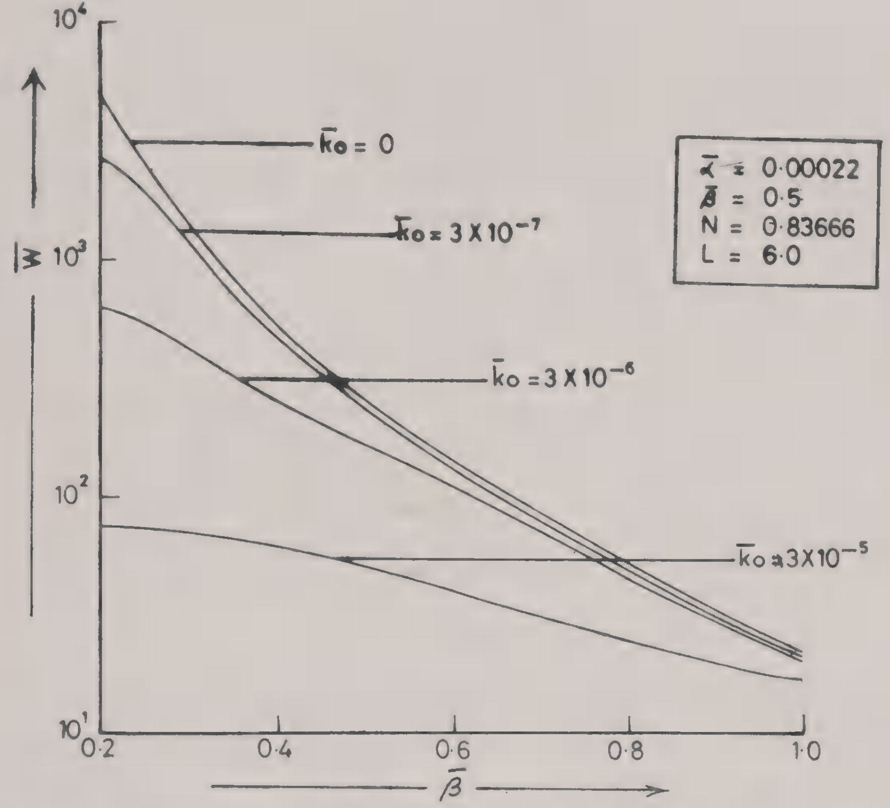

FIG. 4. Variation of non dimensional load carrying capacity $\bar{w}$ with film thickness $\bar{\beta}$ for different values of porosity $\bar{k}_0$.

the increase of interface porosity $\bar{K_0}$, decreases $\overline{W}$. It is interesting to note that by choosing porosity parameter of variation $\bar{\alpha} = .00022$ we have obtained $\bar{p}$ and $\overline{W}$ which are in close agreement with those given by Nigam *et al.*[16]. Thus exponential law of variation of porosity as assumed here is quite efficient to replace a three leyered porous matrix by a continuously varying porosity matrix.

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ON THE REGION FOR LINEAR GROWTH RATE IN ROTATORY HYDROMAGNETIC THERMOHALINE CONVECTION PROBLEM

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The present paper is an improvement of the results of Gupta *et al.*[1] concerning bounds for the linear growth rate of a disturbance in rotatory hydromagnetic thermohaline convection problems of Veronis[2] and Stren's[3] type. As a consequence, the results of Banerjee *et al.*[4] for thermohaline convection and rotatory thermohaline convection problem and of Gupta *et al.*[5] for hydromagnetic thermohaline convection problem are also improved.

1. INTRODUCTION

The thermohaline convection problem has been extensively studied in the recent past under a variety of externally imposed force fields on account of its interesting complexities as a double diffusive phenomenon as well as its direct relevance to many problems of practical interest in the fields of oceanography, astrophysics and chemical engineering etc. The derivation of bounds for the linear growth rate of a disturbance in thermohaline convection problems is an important problem especially when the boundaries are rigid for in this case numerical solutions are most commonly used but they are also laborious. In this situation, derivation of certain integral estimates acquire great importance for they enable one to obtain sufficient conditions of stability and define a possible range of parameters for growing perturbations in case of instability[6]. Baneriee *et al.*[4] and Gupta *et al.*[1,5], investigated in the recent past the thermohaline convection problems and derived bounds for the linear growth rate of an arbitrary oscillatory perturbation, neutral or unstable. However, in deriving these results, some positive definite integrals have been dropped and as a consequence some parameters of the problem which were there in the orginal equations do not occur in the resulting inequalities. Therefore, by making use of suitable estimates for the dropped terms one could possibly derive results which would contain the dropped parameters and improve upon the results of Banerjee *et al.*[4] and Gupta *et al.*[1,5]. The present paper is precisely in this direction.

2. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

The governing nondimensional linearized perturbation equations for Veronis' thermohaline configuration in the presence of a uniform vertical rotation and magnetic field are given by (Gupta *et al.*[1]):

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$$(D^2 - a^2)(D^2 - a^2 - \frac{p}{\sigma}) W + Q\, D (D^2 - a^2) h_z = R_T a^2 \theta - R_S a^2 \Phi + TDZ \quad ...(1)$$

$$(D^2 - a^2 - p)\, \theta = - W, \quad ...(2)$$

$$(D^2 - a^2 - \frac{p}{\tau})\, \Phi = - \frac{W}{\tau} \quad ...(3)$$

$$(D^2 - a^2 - \frac{p\sigma_1}{\sigma})\, h_z = - DW \quad ...(4)$$

$$(D^2 - a^2 - \frac{p}{\sigma})\, Z = - QDX - DW \quad ...(5)$$

$$(D^2 - a^2 - \frac{p\sigma_1}{\sigma})\, X = - DZ \quad ...(6)$$

with the boundary conditions

$$W = 0 = \theta = \Phi = DW = Z = h_z = DX \text{ at } z = 0 \text{ and } z = 1 \quad ...(7)$$

(both the boundaries are rigid and perfectly conducting).

In the above equations the vertical coordinate z has the range $0 \leqslant z \leqslant 1$, $D \equiv d/d_z$, a^2 is the square of the wave number, $p = p_r + ip_i$ is the complex growth rate, σ is the thermal Prandtl number, R_T is the thermal Rayleigh number, R_S is the concentration Rayleigh number, Q is the Chandrasekhar number, T is the Taylor number, τ is the ratio of mass diffusivity to heat diffusivity, σ_1 is the magnetic Prandtl number, W is the vertical velocity, h_z is the vertical magnetic field, θ is the temperature, Φ is the concentration, Z is the vertical vorticity and X is the vertical current density. Further, the governing equations and the boundary conditions for Stern's thermohaline configuration with a uniform vertical rotation and magnetic field are given by eqns. (1)–(7) with $R_T < 0$ and $R_S < 0$.

3. Mathematical Analysis

We prove the following theorems :

Theorem 1— If $(p, W, \theta, \Phi, h_z, Z, X)$, $p = p_r + ip_i$, $p_r \geqslant 0$ $p_i \neq 0$, is a solution of eqns. (1) — (7) then

$$(p_r + \lambda \pi^2)^2 + p_i^2 < \alpha + \lambda^2 \pi^4 \quad ...(8)$$

where

$$\lambda = \min\left(\tau, \frac{\sigma}{\sigma_1}\right) \text{ or } \min(\tau, \sigma) \text{ according as } \sigma_1 \geqslant 1 \text{ or } \sigma_1 < 1 \text{ respectively} \quad ...(9)$$

$$\alpha = \max\left[R_S\sigma, \left(\frac{B + \sqrt{B^2 + 4C}}{2}\right)^2\right] \quad ...(10)$$

$$B = Q\sigma \text{ and } C = T\,\sigma^2 (1 + Q)(1 + Q/\pi^4). \qquad ...(11)$$

PROOF : Multiplying eqn. (1) by W^* (the complex conjugate of W), integrating over the vertical range of z and proceeding as in Gupta *et al.*[1], we have for $p_i \neq 0$

$$\frac{1}{\sigma}\int_0^1 (|DW|^2 + a^2 |W|^2)\, dz + R_T\, a^2 \int_0^1 |\theta|^2 dz + \frac{QT\sigma_1}{\sigma}\int_0^1 |X|^2\, dz$$

$$= \frac{Q\sigma_1}{\sigma}\int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) dz + R_S a^2 \int_0^1 |\phi|^2 dz$$

$$+ \frac{T}{\sigma}\int_0^1 |Z|^2\, dz. \qquad ...(12)$$

Multiplying eqns. (3) – (5) respectively by their complex conjugates, integrating over the range of z by parts repeatedly and utilizing the boundary conditions (7), we get

$$\int_0^1 |(D^2 - a^2)\Phi|^2\, dz + \frac{|p|^2}{\tau^2}\int_0^1 |\Phi|^2\, dz + \frac{2p_r}{\tau}\int_0^1 (|D\Phi|^2$$

$$+ a^2 |\Phi|^2)\, dz = \frac{1}{\tau^2}\int_0^1 |W|^2\, dz \qquad ...(13)$$

$$\int_0^1 |(D^2 - a^2)\, h_z|^2\, dz + \frac{|p|^2 \sigma_1^2}{\sigma^2}\int_0^1 |h_z|^2 dz + \frac{2p_r\, \sigma_1}{\sigma}\int_0^1 (|Dh_z|^2$$

$$+ a^2 |h_z|^2)\, dz = \int_0^1 |DW|^2 dz \qquad ...(14)$$

and

$$\int_0^1 |(D^1 - a^2)\, Z|^2\, dz + \frac{|p|^2}{\sigma^2}\int_0^1 |Z|^2\, dz + \frac{2p_r}{\sigma}\int_0^1 (|DZ|^2$$

$$+ a^2 |Z|^2)\, dz = \int_0^1 |DW|^2\, dz + Q^2 \int_0^1 |DX|^2\, dz$$

$$+ 2Q\, \mathrm{Re}\left(\int_0^1 DX\, DW^*\, dz\right) \qquad ...(15)$$

where 'Re' denotes the real part.

Since $p_r \geqslant 0$, it follows from eqns. (13) — (15) that

$$\frac{|p|^2}{\tau^2}\int_0^1 |\Phi|^2 dz + \frac{2p_r}{\tau}\int_0^1 |D\Phi|^2 dz < \frac{1}{\tau^2}\int_0^1 |W|^2 dz \qquad ...(16)$$

$$\int_0^1 |(D^2 - a^2) h_z|^2 dz < \int_0^1 |DW|^2 dz \qquad ...(17)$$

$$\frac{|p|^2 \sigma_1^2}{\sigma^2}\int_0^1 |h_z|^2 dz + \frac{2p_r\sigma_1}{\sigma}\int_0^1 |Dh_z|^2 dz < \int_0^1 |DW|^2 dz \qquad ...(18)$$

and

$$\frac{|p|^2}{\sigma^2}\int_0^1 |Z|^2 dz + \frac{2p_r}{\sigma}\int_0^1 |DZ|^2 dz < \int_0^1 |DW|^2 dz$$

$$+ Q^2 \int_0^1 |DX|^2 dz + 2Q \operatorname{Re}\left(\int_0^1 DX\, DW^* dz\right). \qquad ...(19)$$

By using the Rayleigh-Ritz inequality[7], we have

$$\int_0^1 |\Phi|^2 dz \leqslant \frac{1}{\pi^2}\int_0^1 |D\Phi|^2 dz \qquad ...(20)$$

since $\Phi(0) = 0 = \Phi(1)$.

Similarly,

$$\int_0^1 |h_z|^2 dz \leqslant \frac{1}{\pi^2}\int_0^1 |Dh_z|^2 dz \qquad ...(21)$$

and

$$\int_0^1 |Z|^2 dz \leqslant \frac{1}{\pi^2}\int_0^1 |DZ|^2 dz. \qquad ...(22)$$

It now follows from inequalities (17), (18) and (19) that

$$\int_0^1 |\Phi|^2 dz < \frac{1}{(|p|^2 + 2p_r\tau\pi^2)}\int_0^1 |W|^2 dz \qquad ...(23)$$

$$\int_0^1 |h_z|^2 dz < \frac{\sigma^2}{(\sigma_1^2 |p|^2 + 2p_r\sigma\sigma_1\pi^2)} \int_0^1 |W|^2 dz, \qquad \ldots(24)$$

and

$$\int_0^1 |Z|^2 dz < \frac{\sigma^2}{(|p|^2 + 2p_r \sigma\pi^2)} \Big[\int_0^1 |DW|^2 dz + Q^2 \int_0^1 |DX|^2 dz^2$$

$$+ 2Q \,\mathrm{Re} \, (\int_0^1 DX \, DW^* dz) \Big]. \qquad \ldots(2.5)$$

Now,

$$\int_0^1 (|Dh_z|^2 + a^2 h_z|^2) dz = -\int_0^1 h_z (D^2 - a^2) h_z^* dz$$

$$\leqslant \int_0^1 |h_z| \; |(D^2 - a^2) h_z^*| \, dz$$

$$\leqslant \{\int_0^1 |h_z|^2 dz\}^{1/2} \; \{\int_0^1 |D^2 - a^2) h_z|^2 dz\}^{1/2}$$

which, upon using inequalities (17) and (24), yields

$$\int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) \, dz < \frac{\sigma}{\sigma_1 \left\{ |p|^2 + \frac{2p_r \sigma\pi^2}{\sigma_1} \right\}^{1/2}} \int_0^1 |DW|^2 dz. \qquad \ldots(26)$$

Multiplying eqn. (16) by X^*, integrating over the range of z by parts, utlizing the boundary conditions (7) and equating the real parts of the resulting equation, we get

$$\int_0^1 (|DX|^2 + a^2 |X|^2 + \frac{p\sigma_1}{\sigma} |X|^2) \, dz = \mathrm{Re} \, (\int_0^1 X^* DZ \, dz)$$

$$= \mathrm{Re} \, (-\int_0^1 ZDX^* dz). \qquad \ldots(27)$$

Similarly, it follows from eqn. (5) that

$$\int_0^1 (|DZ|^2 + a^2 |Z|^2 + \frac{p_r}{\sigma} |Z|^2) \, dz = \mathrm{Re} \, (\int_0^1 Z^* DW \, dz$$

$$+ Q \int_0^1 Z^* DX \, dz) \qquad \ldots(28)$$

Since $p_r \geqslant 0$, it follows from equation (27) that

$$\int_0^1 |DX|^2 dz < \int_0^1 |Z|\,|DX|\; dz$$

$$\leqslant \{\int_0^1 |Z|^2 dz\}^{1/2} \{\int_0^1 |DX|^2 dz\}^{1/2}$$

or

$$\int_0^1 |DX|^2 dz < \int_0^1 |Z|^2 dz$$

from which

$$\int_0^1 |DX|^2 dz < \frac{1}{\pi^2} \int_0^1 |DZ|^2 dz \qquad \ldots(29)$$

can be obtained by using the inequality (22).

Now

$$\int_0^1 Z^* DX\, dz = - \int_0^1 X D Z^* dz$$

which upon using eqn. (6) gives

$$\text{Re}\ (\int_0^1 Z^*\ DX\ dz) = -\int_0^1 (\,|DX|^2 + a^2 |X|^2 + \frac{p_r \sigma_1}{\sigma} |X|^2)dz < 0 \qquad \ldots(30)$$

since $p_r \geqslant 0$.

Equation (28), together with $p_r \geqslant 0$ and the ineqality (30) implies that

$$\int_0^1 |DZ|^2 dz < \text{Re}\ (\int_0^1 Z^* DW\ dz) = \text{Re}\,(-\int_0^1 WDZ^* dz)$$

$$\leqslant \int_0^1 |W|\,|DZ|\, dz$$

$$\leqslant \{\int_0^1 |W|^2 dz\}^{1/2} \{\int_0^1 |DZ|^2 dz\}^{1/2}$$

so that

$$\int_0^1 |DZ|^2 dz < \int_0^1 |W|^2 dz$$

$$\leqslant \frac{1}{\pi^2} \int_0^1 |DW|^2 dz \ \text{(Rayleigh-Ritz inequality).} \qquad \ldots(31)$$

Combining the inequalities (29) and (31), we have

$$\int_0^1 |DX|^2 dz < \frac{1}{\pi^4}\int_0^1 |DW|^2 dz. \qquad ...(32)$$

Further,

$$\text{Re}\left(\int_0^1 DX\, DW^* dz\right) \leqslant \int_0^1 |DX|\ |DW|\, dz$$

$$\leqslant \tfrac{1}{2}\int_0^1 (|DX|^2 + |DW|^2)\, dz. \qquad ...(33)$$

Inequality (25), together with inequalities (32) and (33) gives

$$\int_0^1 |Z|^2 dz < \frac{\sigma^2(1+Q)(1+Q/\pi^4)}{(|p|^2 + 2p_r\sigma\pi^2)}\int_0^1 |DW|^2 dz. \qquad ...(34)$$

Equation (12), together with the integral estimates (23), (26) and (34), gives

$$\left[\frac{1}{\sigma} - \frac{Q}{\left\{|p|^2 + \frac{2p_r\sigma\pi^2}{\sigma_1}\right\}^{1/2}} - \frac{T\sigma(1+Q)(1+Q/\pi^4)}{(|p|^2+2p_r\sigma\pi^2)}\right]\int_0^1 |DW|^2 dz$$

$$+ a^2\left[\frac{1}{\sigma} - \frac{R_S}{(|p|^2+2p_r\tau\pi^2)}\right]\int_0^1 |W|^2 dz$$

$$+ R_T a^2\int_0^1 |\theta|^2 dz + \frac{TQ\sigma_1}{\sigma}\int_0^1 |X|^2 dz < 0. \qquad ...(35)$$

The inequality (35) clearly implies that if $\sigma_1 \geqslant 1$, then

either

$$|p|^2 + 2p_r\tau\pi^2 < R_S\sigma,$$

or

$$\left.|p|^2 + \frac{2p_r\sigma^2\pi}{\sigma_1} < \left(\frac{B+\sqrt{B^2+4C}}{2}\right)^2\right\} \qquad ...(36)$$

where B and C are as given by eqn. (11).

Taking $\lambda = \min\left(\tau, \frac{\sigma}{\sigma_1}\right)$, it follows from (36) that

$$|p|^2 + 2p_r\lambda\pi^2 < \max\left[R_S\sigma, \left(\frac{B+\sqrt{B^2+4C}}{2}\right)^2\right]$$

i. e.

$$(p_r + \lambda\pi^2)^2 + p_i^2 < \alpha + \lambda^2 \pi^4, \qquad \ldots(37)$$

where

$$\alpha = \max\left[R_S\ \sigma, \left(\frac{B + \sqrt{B^2 + 4C}}{2}\right)^2 \right].$$

Similarly, when $\sigma_1 < 1$, we get from the inequality (35) that

$$(p_r + \lambda\pi^2)^2 + p_i^2 < \alpha + \lambda^2 \pi^4 \qquad \ldots(38)$$

where now $\lambda = \min(\tau, \sigma)$.

This completes the proof of the theorem.

Note : It is to be noted that in the absence of rotation ($T = 0$), $\lambda = \min(\tau, \frac{\sigma}{\sigma_1})$, $\forall\ \sigma_1$ and in the absence of magnetic field ($Q = 0$), $\lambda = \min(\tau, \sigma)$.

Theorem 1—(i) shows that the linear growth rate $p\ (= p_r + i\ p_r)$ of an arbitrary oscillatory ($p_i \neq 0$) perturbation, nautral ($p_r = 0$) or unstable ($p_r > 0$), for rotatory hydromagnetic Veronis' thermohaline configuration, must lie inside a semicircle in the right half of the $p_r\ p_i$ -plane, whose centre is $(-\lambda\ \pi^2, 0)$ and (radius)^2 $= \alpha + \lambda^2\ \pi^4$. (ii) It indicates the stabilizing effect of λ since the semicircular region, in which the growth rate p lies, reduces with increasing values of λ. (iii) It gives a more rigid limitation on the growth rate (see Fig. 1) than that given by the semicircular region

$$|\,p\,|^2 = \max\left[R_S\sigma, \left(\frac{B + \sqrt{B^2 + 4C}}{2}\right)^2 \right]$$

of Gupta *et al.*[1].

Special cases of Theorem 1—It follows from Theorem 1 that

(i) for Veronis' thermohaline configuration (VTC, $Q = 0 = T$)

$$(p_r + \tau\pi^2)^2 + p_i^2 < R_S\ \sigma + \tau^2\pi^4.$$

(il) for rotatory VTC ($Q = 0$)

$$(p_r + \lambda\pi^2)^2 + p_i^2 < \alpha + \lambda^2\ \pi^4$$

where

$$\alpha = \max(R_S\ \sigma, T\sigma^2) \text{ and } \lambda = \min(\tau, \sigma),$$

(iii) for hydromagnetic VTC ($T = 0$)

$$(p_r + \lambda\pi^2)^2 + p_i^2 < \alpha + \lambda^2\ \pi^4,$$

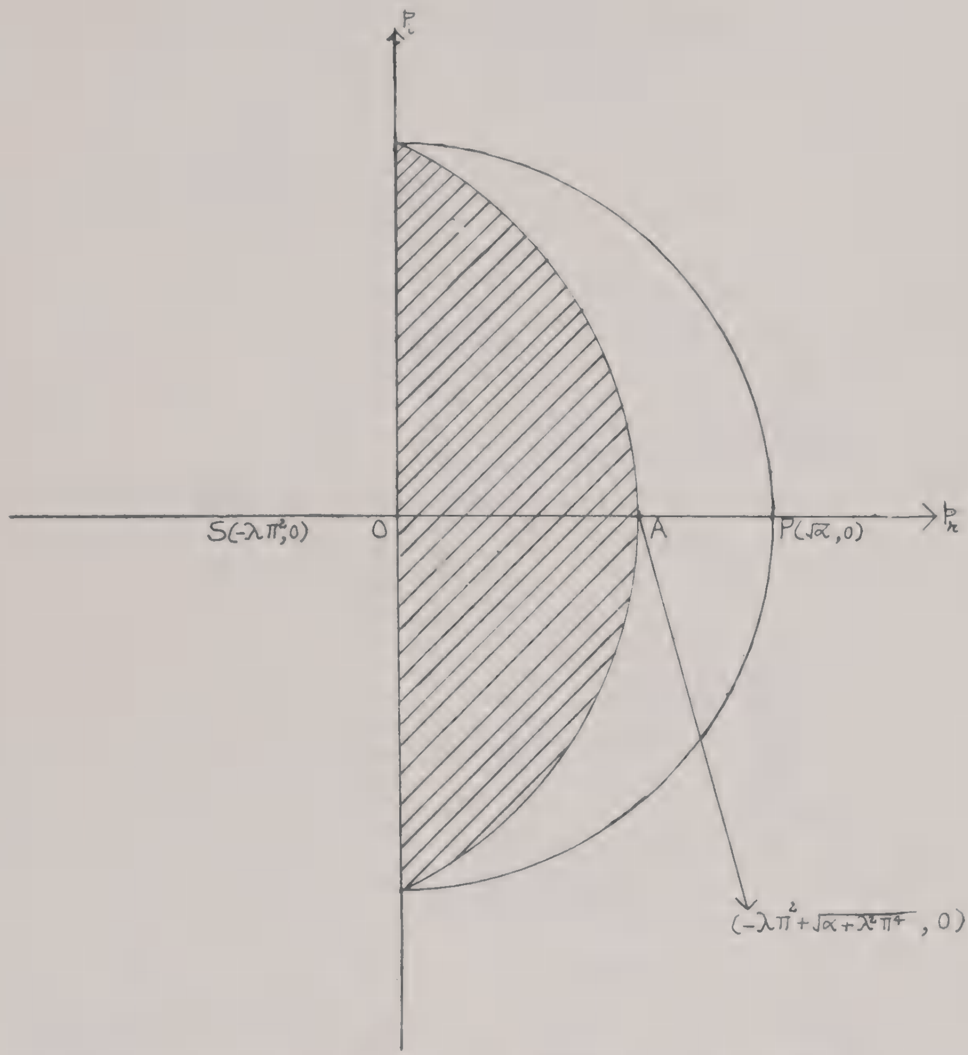

FIG. 1. Shaded area shows the modified region in which the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, for rotatory hydromagnetic Veronis' thermohaline configuration must lie.

Note : $\alpha + \lambda^2\pi^4 < (\sqrt{\alpha} + \lambda\pi^2)^2$, so that $-\lambda\pi^2 + \sqrt{\alpha + \lambda^2\pi^4} < \sqrt{\alpha}$, i. e. OA < OP.

where

$$\alpha = \max\,(R_S\sigma,\ Q^2\sigma^2) \text{ and } \lambda = \min\,(\tau, \frac{\sigma}{\sigma_1}).$$

The above results give more rigid limitations on the linear growth rate of an arbitrary oscillatory perturbation, neutral or unstable, for VTC, rotatory VTC and hydromagnetic VTC than that given by the semicircular regions of Banerjee *et al.*[4] and Gupta *et al.*[5].

Theorem 2—If $(p, W, \theta, \Phi, h_z, Z, X)$, $p = p_r + i\,p_i$, $p_r \geq 0$, $p_i \neq 0$, is a solu-

tion of eqns. (1) – (7) with $R_T = 0$ and $R_S < 0$, then

$$(p_r + \hat{\lambda}\pi^2)^2 + p_i^2 < \hat{\alpha} + \hat{\lambda}^2\pi^4 \qquad \ldots(39)$$

where

$$\hat{\lambda} = \min\left(1, \frac{\sigma}{\sigma_1}\right) \text{ or } \min(1, \sigma) \text{ according as } \sigma_1 \geqslant 1 \text{ or } \sigma_1 < 1 \qquad \ldots(40)$$

respectively,

$$\hat{\alpha} = \max\left[\,|R_T|\,\sigma, \left(\frac{B + \sqrt{B^2 + 4C}}{2}\right)^2\right] \qquad \ldots(41)$$

and B and C are as given in Theorem 1.

PROOF : Putting $R_T = -|R_T|$ and $R_S = -|R_S|$ in eqn. (12), we have

$$\frac{1}{\sigma}\int_0^1 (|DW|^2 + a^2|W^2)\,dz + \frac{QT\sigma_1}{\sigma}\int_0^1 |X|^2\,dz$$

$$+ |R_S|\,a^2\int_0^1 |\Phi|^2\,dz = \frac{Q\sigma_1}{\sigma}\int_0^1 (Dh_z|^2 + a^2|h_z|^2)\,dz$$

$$+ |R_T|\,a^2\int_0^1 |\theta|^2\,dz + \frac{T}{\sigma}\int_0^1 |Z|^2\,dz. \qquad \ldots(42)$$

Using the integral estimates (26), (34) and

$$\int_0^1 |\theta|^2\,dz < \frac{1}{(|p|^2 + 2p_r\pi^2)}\int_0^1 |W|^2 dz \qquad \ldots(43)$$

which is derived in a manner similar to the derivation of (23), we get the result.

Theorem 2—(i) shows that the complex growth rate $p\,(= p_r + i\,p_i)$ of an arbitrary oscillatory perturbation, neutral or unstable, for rotatory hydromagnetic Stern's thermohaline configuration, must lie inside a semicircle, in the right half of the $p_r p_i$-plane, whose centre is $(-\hat{\lambda}\pi^2, 0)$ and $(\text{radius})^2 = \hat{\alpha} + \hat{\lambda}^2\pi^4$. (ii) It gives a more rigid limitation on the growth rate (see Fig. 2 which is essentially Fig. 1 with α, λ replaced by $\hat{\alpha}$ and $\hat{\lambda}$ respectively) then that given by the semicircular region

$$|p|^2 = \max\left[\,|R_T|\,\sigma, \left(\frac{B + \sqrt{B^2 + 4C}}{2}\right)^2\right]$$

of Gupta *et al.*[1].

Special cases of Theorem 2—It follows from Theorem 2 that

(i) for Stern's thermohaline configuration (STC, $Q = 0 = T$)

$$(p_r + \pi^2)^2 + p_i^2 < | R_T | \sigma + \pi^4$$

(ii) for rotatory STC ($Q = 0$)

$$(p_r + \hat{\lambda}\pi^2)^2 + p_i^2 < \hat{\alpha} + \hat{\lambda}^2 \pi^4,$$

where

$$\hat{\alpha} = \max [| R_T | \sigma, T\sigma^2] \text{ and } \hat{\lambda} = \min (1, \sigma).$$

(iii) for hydromagnetic STC ($T = 0$)

$$(p_r + \hat{\lambda}\pi^2)^2 + p_i^2 < \hat{\alpha} + \hat{\lambda}^2 \pi^4$$

where

$$\hat{\alpha} = \max [| R_T | \sigma, Q^2 \sigma^2] \text{ and } \hat{\lambda} = \min (1, \frac{\sigma}{\sigma_1}).$$

The above results give more rigid limitations on the linear growth rate of an arbitrary oscillatory perturbation, neutral or unstable, for STC, rotatory STC and hydromagnetic STC than those given by the semicircular regions of Banerjee *et al.*[3] and Gupta *et al.*[5].

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